On the Hamiltonicity of Product Graph $G \square S_m$, for a Graph $G$ of Order $n$, and Star Graph $S_m$, $n \geq m$

AUREA Z. ROSAL
Polytechnic University of the Philippines
Sta. Mesa, Manila
The Philippines

Abstract:

Given two graphs $G$ and $H$, the Cartesian product, $G \square H$ is the graph whose vertex set is $V(G) \times V(H)$ and the set $\{(u_1, v_1), (u_2, v_2)\}$ is an edge if and only if exactly one of the following is true.

(i) $u_1 = u_2$ and $(v_1, v_2)$ is an edge in $H$.
(ii) $v_1 = v_2$ and $(u_1, u_2)$ is an edge in $G$.

A star graph $S_m$, also known as a complete bipartite graph $K_{1,m}$, is a graph whose vertex set consists of two disjoint sets $V_1 = \{c\}$ and $V_2 = \{v_1, v_2, \ldots, v_m\}$, known as partites, such that no two vertices in $V_2$ are adjacent but all of them are adjacent to $c$. A hamiltonian graph is a graph that contains a cycle containing all its vertices. Clearly, $S_m$ is not hamiltonian for all $m \geq 1$.

In this paper the following shall be proven:

Let $G$ be a hamiltonian graph, $C_n$ be a cycle graph and $K_n$ be a complete graph, all of orders $n$, and $S_m$ be a star graph, $m \geq 1$, then

1. $C_n \square S_m$ is hamiltonian if and only if $n \geq m$, $n \geq 3$
2. $K_n \square S_m$ is hamiltonian if and only if $n \geq m$, $n \geq 2$
3. $G \square S_m$ is hamiltonian if and only if $n \geq m$.

1 Some Preliminaries

For a better understanding of the paper, some terms will be defined.
A graph $\Gamma$ consists of an ordered pair $(V(\Gamma), E(\Gamma))$ where $V(\Gamma)$ is a non-empty set and $E(\Gamma)$ is either a set of two-element subsets of $V(\Gamma)$ or is empty. The elements of $V(\Gamma)$ are called vertices and the elements of $E(\Gamma)$ are called edges. If $\{u, v\}$ is an edge of $\Gamma$ then we say that $u$ and $v$ are adjacent to one another.

The number of vertices of a graph is known as its order and the number of edges is called size. We usually symbolize by $p(\Gamma)$ and $q(\Gamma)$ respectively. The degree of a vertex $u$, $\deg(u)$ is the number of vertices in $\Gamma$ adjacent to $u$.

The complete graph $K_n$ is a graph whose order is $n$ and every distinct vertices are adjacent to one another. Thus, the degree of every vertex in $K_n$ is $n - 1$. The complete bipartite $K_{r,m}$ is a graph whose vertex set $V(K_{r,m})$ is a union of two disjoint sets $V_1$ and $V_2$, known as the partites such that if two vertices are in the same partite then, they are not adjacent. Furthermore, every vertex in one partite is adjacent to every vertex of the other partite. $K_{1,m}$ is also known as star graph and is denoted by $S_m$. Let $V(S_m) = V_1 \cup V_2$, where $V_1 = \{c\}$ and $V_2 = \{v_1, v_2, \ldots, v_m\}$ be the distinct partites of $S_m$, then we shall call $c$ as the center of $S_m$.

A graph may be illustrated as follows: small circles or dots may represent the vertices, and the edges may be represented by lines or curves joining vertices which are adjacent to one another.

![Fig. 1 Complete graph of order 5, $K_5$ and star graph $S_4$](image-url)
2 Cartesian Product of Graphs

Given graphs G and H, a new graph may be formed known as the Cartesian product of G and H written as $G \Box H$. If $\Gamma = G \Box H$, then

$$V(\Gamma) = V(G) \times V(H)$$

and the set \{(u_1, v_1), (u_2, v_2)\} is an edge if and only if exactly one of the following is true:

1. $u_1 = u_2$ and $\{v_1, v_2\}$ is an edge in H, or
2. $v_1 = v_2$ and $\{u_1, u_2\}$ is an edge in G.

Intuitively, the cartesian product $G \Box S_m$ is a graph formed by “replacing” each vertex of $S_m$ with $G$ and edges are formed according to definition.

**Example 2.1.**

Consider $\Gamma = K_2 \Box S_2$.

Let $V(K_2) = \{u_1, u_2\}$ and $V(S_2) = \{v_1, v_2\} \cup \{c\}$. Then,

$$V(\Gamma) = \{(u_1, v_1)(u_1, v_2)(u_1, c)(u_2, v_1)(u_2, v_2)(u_2, c)\}$$

$$E(\Gamma) = \{(u_1, v_1), (u_1, c), (u_1, v_2)(u_1, c), (u_2, v_1), (u_2, c), (u_2, v_2), (u_2, c), (u_1, v_1), (u_2, v_1), (u_1, v_2), (u_2, v_2), (u_1, c), (u_2, c)\}$$

Figure 3 below is the Cartesian product of $K_2 \Box S_2$. One can see that each vertex of $S_2$ was replaced by $K_2$ and corresponding adjacency among vertices were made.
3 Hamiltonian Graph

A path $P_k$ of a graph is a sequence of adjacent vertices $u_1, u_2, \ldots, u_k$ such that no one vertex is repeated. A closed path or cycle is a sequence of adjacent vertices $u_1, u_2, \ldots, u_k, u_{k+1}$ such that $u_1 = u_{k+1}$ and no other vertex is repeated in the sequence. A cycle graph or $n$-cycle $C_n$ is a graph of order $n$ and whose vertices form a cycle.

A graph $H$ is said to be hamiltonian if we can find a cycle in $H$ that contains all its vertices. This cycle is known as a hamiltonian cycle or a spanning cycle. Notice that a complete graph $K_n$, $n \geq 3$ is Hamiltonian while a star graph is not.

It is a known fact in graph theory that the cartesian product of hamiltonian graphs is again hamiltonian. But what about the cartesian product of graphs of which one is not hamiltonian? This paper will prove the following.
Let $G$ be a hamiltonian graph, $C_n$ be a cycle graph and $K_n$ be a complete graph, all of orders $n$, and $S_m$ be a star graph, $m \geq 1$, then

1. $C_n \square S_m$ is hamiltonian if and only if $n \geq m$, $n \geq 3$
2. $K_n \square S_m$ is hamiltonian if and only if $n \geq m$, $n \geq 2$
3. $G \square S_m$ is hamiltonian if and only if $n \geq m$.

**Proposition 3.1.** The cartesian product $C_n S_m$, $m \geq 1$ is hamiltonian if and only if $n \geq m$ for $n \geq 3$.

**Proof.**

Let $\Gamma = C_n S_m$. Without loss of generality, assume $V(C_n) = \{u_1, u_2, \ldots, u_n\}$ and that $u_1, u_2, \ldots, u_n, u_1$ is its cycle. Also, let $V(S_m) = \{c\} \cup \{v_1, v_2, \ldots, v_m\}$, where $c$ is the center of $S_m$. Then:

$$V(\Gamma) = \bigcup_{i=1}^{n} \{(u_i, v_j) | j = 1, 2, \ldots, m\} \cup \{(u_i, c) | i = 1, 2, \ldots, n\}$$

$$E(\Gamma) = \bigcup_{i=1}^{n} \{(u_i, v_j), (u_i, c) | j = 1, 2, \ldots, m\} \cup \left( \bigcup_{i=1}^{n} \{(u_i, v_j), (u_{i+1}, v_j) | j = 1, 2, \ldots, m\} \right)$$

$$\cup \{(u_1, v_j), (u_n, v_j) | j = 1, 2, \ldots, m\} \cup \left( \bigcup_{i=1}^{n} \{(u_i, c), (u_{i+1}, c)\} \right)$$

Suppose $\Gamma$ is hamiltonian, then there exists a cycle $C$ that contains all the vertices of $\Gamma$. Consequently, $C$ will contain a path that will pass through all the vertices of $\Gamma$. Notice that all paths connecting vertices with second coordinates $v_i$ and $v_j$ (distinct), must contain a vertex $(u, c)$ for some $u \in V(C_n)$. Thus, there exists $n - m$ vertices with second coordinate $c$ that will be left “unvisited”, after any path connecting vertices with second coordinates $v_1, v_2, \ldots, v_m$ have been constructed. It follows that $n - m \geq 0$ or $n \geq m$. Since $C_n$ is a cycle, then $n \geq 3$.

Suppose $n \geq m$, and $C_n$ is a cycle, then $n \geq 3$. Also, $m \geq 1$. So, $n - m \geq 2 \geq 1$. Consequently, $n - (m - 1) \geq 3 > 1$. Thus, $1 \leq n - (m - 1) \leq n$. Consider now the following table.
Using the above table as guide, we now form the following cycle:
(u1, v1), (u2, v1), (u3, v1), ..., (un, v1)(un, c), (un, v2),
(u1, v2), (u2, v3), ..., (un−1, v2)(un−1, c)(un−1, v3), (un, v3), (u1, v3), ...
Clearly, above is a spanning cycle of Γ and thus, it is hamiltonian.

We now consider the case of Kn □ Sm. For n = 1, 2, Kn does not contain a cycle. We shall prove however that for n ≥ 2 and n ≥ m, the cartesian product is hamiltonian.

**Proposition 3.2.** The product graph Kn □ Sm , m ≥ 1, is hamiltonian if and only if n ≥ m and n ≥ 2.

**Proof.**
Let V (Kn) = {u1, u2, ..., un} and V (Sm) = {v1, v2, ..., vm} ∪ {c}, c = v_k for all k. Note that for all i = j, ui is adjacent to uj and the elements of {v1, v2, ..., vm} are not adjacent to any vertex in the set but all of them are adjacent to c.

Let Γ = Kn □ Sm be hamiltonian. If n = 1 and m ≥ 1 then K1 is simply a single vertex and K1 □ Sm ~ Sm, which is
not Hamiltonian. This contradicts the assumption that $\Gamma$ is hamiltonian. For $\Gamma = K_2 \square S_1$, it is simply $C_4$. Clearly this is Hamiltonian. Thus, $n \geq 2$. Also, since $\Gamma$ is hamiltonian, then there exists a cycle $C$ that contains all the vertices of $\Gamma$. From proof of previous theorem, it follows that $n \geq m$.

Conversely, let $n \geq 2$, and $n \geq m$. If $n = 2$, then $m = 1, 2$. Now, $K_2 \square S_1$ is $C_4$ and thus hamiltonian. $K_2 \square S_2$ is just the graph

The cycle $(u_1, v_1), (u_1, c), (u_1, v_2), (u_2, v_2), (u_2, c), (u_2, v_1), (u_1, v_1)$ is the required hamiltonian cycle.

For $n \geq 3$, $K_n$ is hamiltonian and thus contains a spanning cycle. From the proof of previous proposition, there exists a spanning cycle for $K_n \square S_m$. It follows that the graph is hamiltonian. Therefore, $K_n \square S_m$ is hamiltonian whenever $n \geq 2$.

–qed–

**Example 3.1.**

Lastly, we prove the more general case for any graph $G$ of order $m$. 

![Fig. 6 Cartesian product of $K_3$ and $S_2$](image-url)
Proposition 3.3. Let $G$ be a hamiltonian graph of order $n$. Then $G \Box S_m$, $m \geq 1$ is hamiltonian if and only if $n \geq m$ and $n \geq 3$.

Proof.
If $G$ is hamiltonian, then $G$ contains a spanning cycle. Hence, its order $n$ must be greater than or equal to 3. We can now apply Proposition 3.1 and thus theorem is proved.

qed

Conclusion

From the propositions presented in this study, one could see that the Cartesian product of two graphs which are not Hamiltonian may be Hamiltonian. Also, that the Cartesian product of a Hamiltonian graph with the star graph is always Hamiltonian.

Recommendation

Currently, work on the Cartesian product of a Hamiltonian graph with $K_{r,s}$ is being studied where neither $r$ nor $s$ is 1.

REFERENCES

[2] Rosal, Aurea: On the Hamiltonicity of $K_n \Box K_{1,m}$, paper presented at the 2011 Annual Convention of MTAP-TL, August 11 - 12, 2011, DLSU Dasmarinas Cavite, Philippines