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Implementation and Development of Differentiation Formula

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Abstract:

Numerical differentiation is the process of finding the numerical value of a derivative of a given function at a given point. In general, numerical differentiation is more difficult than numerical integration. This is because while numerical integration requires only good continuity properties of the function being integrated, numerical differentiation requires more complicated properties such as Lipschitz classes.

There are many applications where derivatives need to be computed numerically. The simplest approach simply uses the definition of the derivative $f'(x) \equiv \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ *for some small numerical value of* $h \ll 1$.

In previous chapters we are developed forward, backward and central difference approximation of first and higher order derivative. Recall that at best these estimate hah error that were o(h^2)- that is their errors were proportional to the square of the step size . We will now illustrate how to develop more accurate formulas by retaining more terms.

Key words: Numerical differentiation, numerical value, derivative

INTRODUCTION

Consider a function of a single variable y=f(x). If f(x) is defined as an expression, its derivative may often be determined using the techniques of calculus. However, when f(x) is a complicated function or when it is given in a form, we use numerical method .Here, we will discuss numerical method for approximating the derivatives $f^{(r)}(x) r \geq 1$ of a given function f(x) .The accuracy attainable by these method would depend on the given function and the order of the polynomial used . If the polynomial fitted is exact then the error would be theoretically zero. In practice, however, rounding errors will introduce errors in the calculated values.

NUMERICAL DIFFERENTIATION:

In the case of numerical data, the function form of f(x) is not known in general. First we have to find an appropriate form of f(x) and then obtain its derivatives. So "Numerical differentiation" is concerned with the method of finding the successive derivatives of a function at a given argument, using the given table of entries corresponding to a set of arguments, equally or unequally spaced. Using the theory or interpolation, a suitable interpolation polynomial can be chosen to represent the function to good degree of approximation in the given interval of the argument. For the proper choice of interpolation formula, the criterion is same as in case of interpolation problem, In case of equidistant values of x, if the derivative is to be found at a point near the beginning or the end given set of values, we should use Newton's forward near the middle of the give set of values, then we should use any one of the central difference formula. However, if the function is not known at equidistant values of x, we shall use Newton's divided difference or Lagrange's formula.

METHODOLOGY

Formula for Derivatives:

1-Newton's forward difference formula is

$Y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_o + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_o + \cdots$ (1)

$u = \frac{x-a}{h}$ (2)

where

differentiating eqn.(1)with respect to u, we get

$\frac{dy}{du} = \Delta y_o + \frac{2u-1}{2}\Delta^2 y_o + \frac{3u^2-6u+2}{6}\Delta^3 y_o$ (3)

differentiating eqn. (2) with respect to x, we get

$\frac{du}{dx} = \frac{1}{h}$ (4)

We known that

$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{1}{h}[\Delta y_o + \frac{2u-1}{2}\Delta^2 y_o + \frac{3u^2-6u+2}{6}\Delta^3 y_o]$ (5)

Expression (5) provides us the value of $\frac{dy}{dx}$ at any which is not tabulated.

Formula (5) becomes simple for tabulated values of x, in particular when x=a and u=o putting u=o in(5), we get

$(\frac{dy}{dx})_{\chi=a} = \frac{1}{h}[\Delta y_o - \frac{1}{2}\Delta^2 y_o + \frac{1}{3}\Delta^3 y_o - \frac{1}{4}\Delta^4 y_o + \cdots..]$ (6)

differentiating eqn.(5)with respect to x, we get

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dx}$

$= \frac{1}{h}[\Delta^2 y_o + (u-1)\Delta^3 y_o + (\frac{6u^2-18u+11}{12})\Delta^4 y_o \ldots]\frac{1}{h}$

$= \frac{1}{h^2}[\Delta^2 y_o + (u-1)\Delta^3 y_o + (\frac{6u^2-18u+11}{12})\Delta^4 y_o \ldots]$….(7)

Putting u=0 in eqn.(7), we get

$(\frac{d^2u}{dx^2})_{\chi=a} = \frac{1}{h^2}(\Delta^2 y_o - \Delta^3 y_o + \frac{11}{12}\Delta^4 y_o - \cdots)$ (8)

Similarly, we get

$(\frac{d^3y}{dx^3})_{\chi=a} = \frac{1}{h^3}(\Delta^3 y_o - \frac{3}{2}\Delta^4 y_o + \cdots)$ (9)

And so on.

Formulae for computing higher derivatives may be obtained by successive differentiation.

Aliter: We known that

$E = e^{hd} \rightarrow 1+\Delta = e^{hd}$

hd =log (1+Δ)=$\Delta-\frac{\Delta^2}{2}+\frac{\Delta^3}{3}-\frac{\Delta^4}{4}+\cdots$

D=$\frac{1}{h}(\Delta-\frac{\Delta^2}{2}+\frac{\Delta^3}{3}-\frac{\Delta^4}{4}+\cdots)^2$

Similarly, $D^2=\frac{1}{h^2}(\Delta-\frac{\Delta^2}{2}+\frac{\Delta^3}{3}-\frac{\Delta^4}{4}+\cdots)^2 = \frac{1}{h^2}(\Delta^2-\Delta^3+\frac{11}{12}\Delta^4-\cdots)$

And $D^3=\frac{1}{h^3}(\Delta^3-\frac{3}{2}\Delta^4+\cdots)$

2-Newton's backward difference formula is

$Y=y_n+u\nabla y_n+\frac{u(u+1)}{2!}\nabla^2 y_n+\frac{u(u+1)(u+2)}{3!}\nabla^3 y_n+$………(10)

Where $u=\frac{x-xn}{h}$ (11)

differentiating eqn.(10)with respect to u, we get

$\frac{dy}{du}=\nabla y_n+(\frac{2u+1}{2})\nabla^2 y_n+(\frac{3u^2-6u+2}{6})\nabla^3 y_n+$……….(12)

differentiating eqn.(11)with respect to x , we get

$$\frac{du}{dx}=\frac{1}{h}$$

Now, $\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}$……(13)

$=\frac{1}{h}[\nabla y_n+(\frac{2u+1}{2})\nabla^2 y_n+(\frac{3u^2-6u+2}{6})\nabla^3 y_n+\ldots\ldots\ldots.]$(14)

Expression (14) provides us the value of $\frac{dy}{dx}$ at any x which is not tabulated.

Atx= x_n , we have u=0

Putting u=0 in eqn. (14) , we get

$(\frac{dy}{dx})_{x=xn} =\frac{1}{h}[\nabla y_n+\frac{1}{2}\nabla^2 y_n+\frac{1}{3}\nabla^3 y_n+\ldots\ldots\ldots.]$…..(15)

differentiating eqn.(14)with respect to x , we get

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dx}$$

$=\frac{1}{h^2}[\nabla^2 y_n+(u+1)\nabla^3 y_n+\frac{6u^2+18u+11}{12}\nabla^4 y_n]$..(16)

Putting u=0 in eqn. (16) , we get

$(\frac{d^2y}{dx^2})_{x=xn} =\frac{1}{h^2}[\nabla^2 y_n+\nabla^3 y_n+\frac{11}{12}\nabla^4 y_n \ldots\ldots\ldots.]$…..(17)

Similarly,

$(\frac{d^3y}{dx^3})_{x=xn} =\frac{1}{h^3}[\nabla^3 y_n+\frac{3}{2}\nabla^4 y_n \ldots\ldots\ldots.]$…..(18)

Formulae for computing higher derivatives may be obtained by successive differentiation.

Aliter : We known that

$E^{-1}=1-\nabla$

$e^{-hd}=1-\nabla$

$-hd=\log(1-v)=-[\nabla+\frac{1}{2}\nabla^2 y_n+\frac{1}{3}\nabla^3 y_n+\frac{1}{4}\nabla^4 y_n+\ldots]$

$\Rightarrow D=\frac{1}{h}[\nabla+\frac{1}{2}\nabla^2 y_n+\frac{1}{3}\nabla^3 y_n+\frac{1}{4}\nabla^4 y_n+\ldots]$

Also, $D^2=\frac{1}{h^2}[\nabla+\frac{1}{2}\nabla^2 y_n+\frac{1}{3}\nabla^3 y_n+\frac{1}{4}\nabla^4 y_n+\ldots]^2$

Similarly,

$D^3=\frac{1}{h^3}[\nabla^3 y_n+\frac{3}{2}\nabla^4 y_n+\ldots]$ and so on.

3-Stirling's central difference interpolation formula is

$$Y=y_0+\frac{u}{1!}\left(\frac{\Delta y_0+\Delta y_{-1}}{2}\right)+\frac{u^2}{2!}\Delta^2 y_{-1}+\frac{u(u^2-1^2)}{3!}\left(\frac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{2}\right)+\frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} \quad (19)$$

Where $u=\frac{x-a}{h}$………(20)

Differentiating eqn.(19)with respect to u , we get

$$\frac{dy}{du}=\left(\frac{\Delta y_0+\Delta y_{-1}}{2}\right)+u\Delta^2 y_{-1}+\frac{(3u^2-1)}{6}\left(\frac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{2}\right)+\frac{(4u^3-2u)}{4!}\Delta^4 y_{-2}+\cdots(21)$$

Differentiating eqn.(20)with respect tox , we get

$$\frac{du}{dx}=\frac{1}{h} \quad (22)$$

Now , $\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}$

$$=\frac{1}{h}\left[\left(\frac{\Delta y_0+\Delta y_{-1}}{2}\right)+u\Delta^2 y_{-1}+\frac{(3u^2-1)}{6}\left(\frac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{2}\right)+\frac{(4u^3-2u)}{4!}\Delta^4 y_{-2}+\cdots(23)\right.$$

Expression (23) provides us the value of $\frac{dy}{dx}$ at any x which is not tabulated.

Put x=a, we have u=0

Putting u=0 in eqn.(23) , we get

$(\frac{dy}{dx})_{\chi=a} = \frac{1}{h}[(\frac{\Delta y_o+\Delta y_{-1}}{2}) - \frac{1}{6}(\frac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{2}) + \frac{1}{30}\frac{\Delta^5 y_{-2}+\Delta^5 y_{-3}}{2} -]$

(24)

Differentiating eqn.(23)with respect to x , we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dx}$$

$=\frac{1}{h^2}[(\Delta^2 y_{-1} + u(\frac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{2}) + \frac{(6u^2-1)}{12}\Delta^4 y_{-2} + (\frac{2u^3-3u}{12})\frac{\Delta^5 y_{-2}+\Delta^5 y_{-3}}{2} +]$

(25)

Putting u=0 in eqn.(24) , we get

$(\frac{d^2y}{dx^2})_{\chi=a} = \frac{1}{h^2}(\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} - \cdots)$ (26)

And so on.

Formulae for computing higher derivatives may be obtained by successive differentiation.

RESULT AND DISCUSSION

We have already introduced the notion of numerical differentiation in previous chapters and we employed the Taylors series expansion to derive the finite divided difference approximation of derivatives. In previous chapters we are developed forward, backward and central difference approximation of first and higher order derivative .Recall that at best these estimate hah error that were o(h^2)- that is ,their errors were proportional to the square of the step size .This level of accuracy is due to the number of terms of Taylor's that were retained during the derivation of these formulas . We will now illustrate how to develop more accurate formulas by retaining more terms.

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