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Some Bounds of Rainbow Edge Domination in Graphs

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Abstract:

The edge dominating set of a graph G = (V, E) is the subset F ⊆ E such that each edge in E is either in F or is adjacent to an edge in F. The maximum degree of an edge in G is defined as $\Delta'(G)$ *and diameter of a graph is the length of shortest path between the most distanced nodes. In this paper we try to find some bounds for the rainbow edge domination number of a graph in terms of maximum edge degree* $\Delta'(G)$ *and the diameter of the graph.*

Key words: Diameter, 2rainbow edge domination, 2rainbow edge domatic, k-rainbow edge dominating family

1. Introduction

The dominating set of a graph G = (V, E) is the subset S⊆ V such that every vertex v ∈V is either an element of S or is adjacent to some element of S. A dominating set S is a minimal dominating set if no proper subset S′ ⊂ S is a dominating set. The cardinality of minimal dominating set of G is called domination number of G which is denoted by $\gamma(G)$.The open neighborhood N(v) of v ∈V(G) is the set of vertices adjacent to v and the set N[v]=N(v)∪ {v} is the closed neighborhood of v. For any number "n" ,⌈n⌉ denotes the smallest integer not less than "n" and ⌊n⌋ denotes the greatest integer not greater than "n".

An edge "e" of a graph G is said to be incident with the vertex v if v is an end vertex of e . Two edges e and f which incident with a common vertex v are said to be adjacent. A subset F ⊆ E is an edge dominating set if each edge in E is either in F or is adjacent to an edge in F. An edge dominating set F is called minimal if no proper subset F′ of F is an edge dominating set.

The edge domination number $\gamma'(G)$ is the cardinality of minimal edge dominating set. The open neighborhood of an edge $e \in$ E is denoted as N(e) and it is the set of all edges adjacent to e in G, further N [e]=N(e)∪{e} is the closed neighborhood of " e " in G. For all terminology and notations related to graph theory not given here we follow [7]. The motivation of domination parameters are obtained from [7] and [8]. This work is mainly based on [2], [3], [5] and [6].

2. 2-Rainbow edge domination function

Let G=(V,E) be a graph and let g be a function that assigns to each edge a set of colors chosen from the power set of {1,2} i.e., g:E(G)→$\mathcal{P}$ {1,2}. If for each edge e ∈E(G) such that g(e) = ϕ,we have $\bigcup_{f\in N(e)} g(f)$={1,2},then g is called 2-Rainbow edge domination function(2REDF) and the weight w(g) of a function is defined as w(g) = $\sum_{f\in E(G)} |g(f)|$.

The minimum weight of 2REDF is called 2-rainbow edge domination number (2REDN) of G denoted by γ'_{r2} (G).

3. Roman domination function

A Roman dominating function on a graph G = (V,E) is a function f : V → {0, 1, 2} satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of Roman dominating function is the value $f(v) = \sum_{u\in V}(f(u))$. The minimum weight of a Roman dominating function on a graph G is called the Roman domination number of G and denote by γ'_R (G).

Theorem 3.1 For any graph G, γ'(G) ≤ γ'_{r2}(G) ≤ γ'_R (G) ≤ $2\gamma'$(G) .

Proof From the theorem in [1] we have γ'(G) ≤ γ'_R (G) ≤ $2\gamma'$(G) so to prove the theorem we need to prove first γ'_{r2}(G) ≤ γ'_R (G) and also γ'(G) ≤ γ'_{r2}(G).

Let f : E(G) → $\mathcal{P}\{0,1,2\}$ be Roman edge dominating function with minimum weight it means W(f) = γ'_R (G). Now we can define a function g : E(G) → $\mathcal{P}\{1,2\}$ as the following ;

$$g(e) = \begin{cases} \emptyset & \text{if} \quad e \in E_0 \\ \{1\} or \{2\} & \text{if} \quad e \in E_1 \\ \{1,2\} & \text{if} \quad e \in E_2 \end{cases}$$

We assign $\emptyset$ for any edge $e \in E_0$ it means g(e) = $\emptyset$ when $e \in E_0$ and $e \in E_0$ means f (e) = 0 and since f : E(G) → $\mathcal{P}\{0,1,2\}$ is γ'_R –function, then any edge $e \in E_0$ must be adjacent to edge h ∈ E_2, i.e.,f (h) = 2 and in the function g : E(G) → $\mathcal{P}\{1,2\}$, if $h \in E_0$ then g(h) = {1,2}. Hence g : E(G) → $\mathcal{P}\{1,2\}$ is a 2-rainbow edge domination function in G with the weight W(g) that means γ'_{r2}(G) ≤ W(g) (1)

But by the definition of the function g : E(G) → $\mathcal{P}\{1,2\}$, we can get that

W(g) = $|E_1|+|E_2|$ = W(f) = γ'_R (G) (2)

By (1) and (2) we get

γ'_{r2}(G) ≤ γ'_R (G) (3)

For the lower bound, let g : E(G) → $\mathcal{P}\{1,2\}$ be a 2-rainbow dominating function with the minimum weight i.e., W(g) = γ'_{r2}(G).

By the function g: E(G) → $\mathcal{P}\{1,2\}$ the edges of G can be partition to four sets as the following;

$E_0 = e_i \in E(G); g(e_i) = \emptyset \ , i = 1,2,\dots,n\}$

${}^1E_1 = \{ e_i \in E(G); g(e_i) = \{1\} \ , i = 1,2,\dots,n\}$

${}^2E_1 = \{e_i \in E(G); g(e_i) = \{2\} \ , i = 1,2,\dots,n\}$

$E_2 = \{ e_i \in E(G); g(e_i) = \{1,2\} \ , i = 1,2,\dots,n\}$

For the edge domination we can define the edge domination in G as following;

Let G = (V,E) be a graph. An edge dominating function of G is a function f : E(G) → {0,1} such that for any edge $e \in$ E(G) for which $f(e) = 0$ is adjacent to at least one edge h for which $f(h) = 1$. The weight of an edge dominating function is the value $f(E) = \sum_{e\in E(G)} f(e)$. The edge domination number of G denoted by $\gamma'(G)$ is the minimum weight of an edge dominating function in G.

Now let f : E(G) → {0,1} be define as

$$f(e) = \begin{cases} 0 & if \quad e \in E_0 \\ 1 & otherwise \end{cases}$$

It is obvious that f : E(G) → {0,1} is an edge dominating function in G and

$W(f) = |{}^1E_1| + |{}^2E_1| + |E_2| \le |{}^1E_1| + |{}^2E_1| + 2\,|E_2| = \gamma'_{r2}(G)$

Therefore;

$\gamma'(G) \le W(f) \le \gamma'_{r2}(G)$

Hence

$$\gamma'(G) \le \gamma'_{r2}(G) \qquad (4)$$

Hence from (3) and (4) we have

$\gamma'(G) \le \gamma'_{r2}(G) \le \gamma'_R(G) \le 2\gamma'(G)$.

Corollary 3.2 Let G = (V,E) be a graph, $\gamma'_{r2}(G) = \gamma'_R(G) = \gamma'(G) = 1$ if and only if $G \cong mK_2$ for $m \ge 1$.

Proof If $\gamma'_{r2}(G) = \gamma'_R(G) = \gamma'(G) = 1$, clearly if $\gamma'_R(G) = 1$ then $V_2 = \emptyset$ and $V_0 = \emptyset$ so $|V_1| = 1$ that means there is one edge on G, that means there is only one case for $\gamma'_R(G) = 1$ hence $\gamma'_R(G) = 1$ if and only if $G \cong mK_2$. Conversely it is clear that if $G \cong mK_2$ then $\gamma'_{r2}(G) = \gamma'_R(G) = \gamma'(G) = 1$. □

We know that $\gamma'_{r2}(G) = \gamma_{r2}(L(G))$ for any graph G where L(G) is the line graph of G. To study when $\gamma'_{r2}(G) = \gamma_{r2}(G)$ we have two cases either G ≅ L(G) and in this case G ≅ kC_n for any positive integers k and n.

Observation 3.3 For any graph G ≅ kC_n we have $\gamma'_{r2}(G) = \gamma_{r2}(G)$.

Proposition 3.4 For any path P_n where $n \geq 2$, $\gamma'_{r2}(G) = \gamma'_R(G)$ if and only if $n = 2,3,5$ or 7 i.e., G ≅ P_2, P_3, P_5 or P_7 .

Proof Let G ≅ P_n then we have $\gamma'_R(G) = \left\lfloor \frac{2n}{3} \right\rfloor$. From theorem

$$\gamma'_{r2}(P_n) = \begin{cases} \frac{n}{2} & if \quad n \text{ is even} \\ \frac{n+1}{2} & if \quad n \text{ is odd} \end{cases}$$

we can write $\left\lfloor \frac{2n}{3} \right\rfloor$ as the following;

$$\gamma'_R(P_n) = \left\lfloor \frac{2n}{3} \right\rfloor = \begin{cases} \frac{2n}{3} & if \quad n \equiv 0(mod3) \\ \frac{2n-2}{3} & if \quad n \equiv 1(mod3) \\ \frac{2n-1}{3} & if \quad n \equiv 2(mod3) \end{cases}$$

Case 1 If $\gamma'_{r2}(G) = \frac{n}{2}$ then only n should be equal to 2 to get $\gamma'_{r2}(G) = \frac{n}{2} = \frac{2n-1}{3} = \gamma'_R(G)$ hence G ≅ P_2.

Case 2 $\gamma'_{r2}(G) = \frac{n+1}{2}$ then

either $\frac{n+1}{2} = \frac{2n}{3}$ then $n = 3$

or $\frac{n+1}{2} = \frac{2n-2}{3}$ then n = 7

or $\frac{n+1}{2} = \frac{2n-1}{3}$ then n = 5

Hence G ≅ P_3 or P_5 or P_7. conversely, if G ≅ P_2 or P_3 or P_5 or P_7 then $\gamma'_{r2}(G) = \gamma_{r2}(G)$. □

Proposition 3.5 for any path P_n with odd number of vertices $\gamma'_{r2}(P_n) = \gamma'_R(P_n)$.

Proof By using theorems

$$\gamma'_{r2}(P_n) = \begin{cases} \frac{n}{2} & if \quad n \text{ is even} \\ \frac{n+1}{2} & if \quad n \text{ is odd} \end{cases}$$

And $\gamma_{r2}(P_n) = \left\lfloor\frac{n}{2}\right\rfloor+1$

We can write

$$\gamma_{r2}(P_n) = \begin{cases} \frac{n+2}{2} & if \quad n \text{ is even} \\ \frac{n+1}{2} & if \quad n \text{ is odd} \end{cases}$$

Then $\frac{n+2}{2}$ cannot be equal to $\frac{n}{2}$. Hence $\gamma'_{r2}(P_n) = \gamma_{r2}(P_n) = \frac{n+1}{2}$ if n is odd. □

Theorem 3.6 Let G be a connected graph with q edges and contains one edge e_0 with degree deg(e_0) = q - γ'(G). Then γ'_{r2}(G) either is equal to γ'(G) +1 or γ'(G).

Proof Let G = (V, E) be connected graph with q edges, let $e_0 \in$ E(G) such that deg(e_0) = q - γ'(G). Now let f: E(G) → $\mathcal{P}\{0,1\}$ defined as following;

$$f(e) = \begin{cases} \{1,2\} & \text{if} \quad e = e_0 \\ \{1\}\text{or}\{2\} & \text{if} \quad e \in \text{E} - \text{N}[e_0] \\ \emptyset & \text{if} \quad e \in \text{N}[e_0] \end{cases}$$

clearly f : E(G) → $\mathcal{P}\{1,2\}$ is a 2-Rainbow edge dominating function in G and the weight of f is W(f) = 2 + q – (q -γ'(G) +1) = γ'(G) +1

Therefore

γ'_{r2}(G) ≤ γ'(G) +1 (1)

And by theorem 3.1 we have

γ'(G) ≤ γ'_{r2}(G) (2)

By (1) and (2) γ'_{r2}(G) has two values either γ'(G) or γ'(G) +1. □

Theorem 3.7 Let G = (V,E) be a graph and let f : E(G) → $\mathcal{P}\{1,2\}$ be its 2-Rainbow edge domination function such that $|{}^1E_1| = 0$. Then $\langle {}^2E_1 \rangle \cong sK_2 \cup tP_3$ for some integers s,t≥ 0 **.**

Proof Let f : E(G) → $\mathcal{P}\{1,2\}$ be a 2-Rainbow edge dominating function with the minimum weight in G that means W(f) = γ'_{r2}(G) and this function has the property $|{}^1E_1| = 0$. Now to prove that $\langle {}^2E_1 \rangle \cong sK_2 \cup tP_3$ it is enough to prove that no edge in $^2 E_1$ has degree more than two. Suppose there is some edge in $^2 E_1$ of degree more than two then let e_1 and e_2 and e_3 be the sequence edge of P_4 in $\langle ^2 E_1 \rangle$ or the edges of $K_{1,3}$ in $\langle ^2 E_1 \rangle$. Clearly $f(e_1) = f(e_2) = f(e_3) = 2$. Let us define the function $\acute{f}$: E(G) → $\mathcal{P}\{1,2\}$ as

$$\acute{f}(e) = \begin{cases} \emptyset & \text{if} \quad e \in \{e_1, e_3\} \\ \{1,2\} & \text{if} \quad e_1 = e_2 \\ f(e) & \text{otherwise} \end{cases}$$

It is easy to see that $\acute{f}$ is 2-Rainbow edge dominating function in G and W($\acute{f}$) = W(f) – 1 hence W($\acute{f}$) ≤ W(f) this is contradiction for the definition of 2-Rainbow edge domination in graph. Therefore there is no edge of degree more than two in $\langle ^2 E_1 \rangle$ that means either $\langle ^2 E_1 \rangle$ is P_3 or K_2 or disjoint union of P_3 and K_2. Hence $\langle ^2 E_1 \rangle \cong sK_2 \cup tP_3$. □

Note similarly in theorem if $|^2E_1| = 0$ then we can prove in the same way that $\langle ^1 E_1 \rangle \cong sK_2 \cup tP_3$.

Proposition 3.8 For any graph G if there exists 2-Rainbow edge dominating function f: E(G) → $\mathcal{P}\{1,2\}$ such that either $|^1E_1| = 0$ or $|^2E_1| = 0$ then $\gamma'_{r2}(G) = \gamma'_R(G)$.

Proof Let G = (V,E) be a graph and let f : E(G) → $\mathcal{P}\{1,2\}$ be a 2-Rainbow edge dominating function in G and without lose of generality let $|^1E_1| = 0$. Then for any edge e in G with $f(e) = \emptyset$ there existat least one edge $\acute{e}$ adjacent to e such that $f(\acute{e}) = \{1,2\}$.

Now let g: E(G) → {0,1,2} defined as

$$g(e) = \begin{cases} 0 & \text{if} \quad e \in E_0 \\ 1 & \text{if} \quad e \in {}^2 E_1 \\ 2 & \text{if} \quad e \in E_2 \end{cases}$$

Then clearly g is Roman edge dominating function and W(g) = $|{}^2E_1|$ +2 $|E_2|$

Therefore

γ'_R (G) ≤ $|{}^2E_1|$ +2 $|E_2|$ = W(f) = γ'_{r2}(G)

Hence

γ'_R (G) ≤ γ'_{r2}(G) (1)

Also by the theorem 3.1, we have

γ'_{r2}(G) ≤ γ'_R (G) (2)

From (1) and (2) we have γ'_{r2}(G) = γ'_R (G). similarly if $|{}^2E_1| = 0$ we can prove in the same way that γ'_{r2}(G) = γ'_R (G). □

Theorem 3.9 Let G be a graph and let f : E(G) → $\mathcal{P}\{1,2\}$ is 2-Rainbow edge dominating function in G. Then

i) There exist no common end vertex between the edges in E_2 the edges in ${}^1E_1 \cup {}^2E_1$.

ii) If one of ${}^1E_1$ or ${}^2E_1$ be equal to zero then E_2 is minimum edge dominating set of induced subgraph $\langle E_2 \cup E_0 \rangle$.

iii) Each edge in the set E_0 is adjacent to at most two edges of ${}^1E_1 \cup {}^2E_1$.

Proof

i) Let e and $\acute{e}$ be any two edges in G such that f (e) ={1,2} and f ($\acute{e}$) = {1} or {2}, let $e = uv$ and $\acute{e} = vw$ that means $e \in E_2$ and $\acute{e} \in E_1 \cup E_2$ and e and $\acute{e}$ has common vertex v. Now we can define the function $\acute{f}$: E(G) → $\mathcal{P}\{1,2\}$ as the following;

$$\acute{f}(h) = \begin{cases} \emptyset & \text{if } \quad h = \acute{e} \\ f(h) & \text{otherwise} \end{cases}$$

It is easy to see that $\acute{f}$: E(G) → $\mathcal{P}\{1,2\}$ is a 2-Rainbow edge dominating function in G and W($\acute{f}$) = W(f) − 1 and this is contradiction with the fact that f : E(G) → $\mathcal{P}\{1,2\}$ is 2-Rainbow edge dominating function. Hence there is no common end vertex between any edge in E_2 and any edge in ${}^1E_1 \cup {}^2E_1$.

ii) Let D be a dominating set of induced subgraph $\langle E_2 \cup E_0 \rangle$ and let $|D| < |E_2|$ and let $|{}^1E_1| = 0$. We can define the function f : E(G) → $\mathcal{P}\{1,2\}$ as

$$g(h) = \begin{cases} \{1,2\} & \text{if} \quad h \in D \\ \{2\} & \text{if} \quad h \in {}^2E_1 \\ \emptyset & \text{if} \quad h \in (E_2 \cup E_0) - D \end{cases}$$

Then obviously g is 2-Rainbow edge dominating function in G and $W(g) = |{}^1E_1| + 2|D| < |E_1| + 2|E_2| = W(f) = \gamma'_{r2}(G)$ and this is contradiction.

Hence $|E_2|$ is the minimum dominating set of the induced subgraph $\langle E_2 \cup E_0 \rangle$.

iii) Suppose that $e_0 \in E_0$ is adjacent to three edges in ${}^1E_1 \cup {}^2E_1$ (say) e_1, e_2 and e_3 . let g: E(G) → $\mathcal{P}\{1,2\}$ defined as

$$g(e) = \begin{cases} \{1,2\} & \text{if} \quad e \in E_2 \cup \{e_0\} \\ \{1\} \text{ or } \{2\} & \text{if} \quad e \in ({}^1E_1 \cup {}^2E_2) - \{e_1, e_2, e_3\} \\ \emptyset & \text{if} \quad e \in E_0 \cup \{e_1, e_2, e_3\} - \{e_0\} \end{cases}$$

clearly any edge assign to $\emptyset$ by g is adjacent to edge h such that $g(h) = \{1,2\}$ or adjacent to two edges h', h'' such that $g(h') = \{1\}$ and $g(h'') = \{2\}$. Therefore g : E(G) → $\mathcal{P}\{1,2\}$ is 2-Rainbow edge dominating function in G and $W(g) = |{}^1E_1| + |{}^2E_1| - 3 + 2|E_2| + 2 = |{}^1E_1| + |{}^2E_1| + 2|E_2| - 1 < |{}^1E_1| + |{}^2E_1| + 2|E_2| = W(f) = \gamma'_{r2}(G)$ which is contradiction. Hence each edge in E_0 is adjacent to at most two edges of ${}^1E_1 \cup {}^2E_1$. □

Proposition 3.10 Let G = (V, E) be a graph with $q \geq 2$ edges and contains at least one edge of degree $q-1$. Then $\gamma'(G) = 1$ and $\gamma'_{r2}(G) = 2$.

Proof Let G be a graph with q edges and let $e_0 \in E(G)$ such that $\deg(e_0) = q-1$. It is clear that e_0 will dominate all the edges in G therefore $\gamma'(G) = 1$.

Now let g : E(G) → $\mathcal{P}\{1,2\}$ be a function defined as

$f(e) = \begin{cases} \{1,2\} & \text{if } \quad e = e_0 \\ \emptyset & \text{otherwise} \end{cases}$

Since $\deg(e_0) = q - 1$, then g is 2-Rainbow edge dominating function in G, and W(g) = 2.

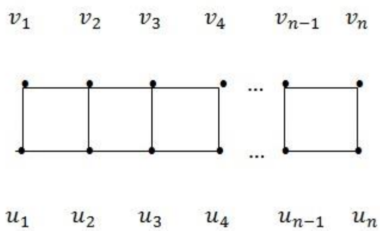
Hence γ'_{r2}(G) ≤ 2.

Hence γ'_{r2}(G) = 1 or 2.

But from theorem γ'_{r2}(G) = 1 if and only if G≅ K_2 hence γ'_{r2}(G) = 2. □

Proposition 3.11 Let G ≅ $K_2 \square P_n$ then γ'_{r2}(G) = n.

Proof Let G ≅ $K_2 \square P_n$ as the following figure



We define the function f: E(G) → $\mathcal{P}\{1,2\}$ as;

$$f(e) = \begin{cases} \{1\} & \text{if} \quad e = v_{2k}u_{2k} \quad k \geq 1 \\ \{2\} & \text{if} \quad e = v_{2k-1}u_{2k-1} \quad k \geq 1 \\ \emptyset & \text{otherwise} \end{cases}$$

if f : E(G) → $\mathcal{P}\{1,2\}$, clearly every edge e with $f(e) = \emptyset$ has two neighborhood edges e' and e'' such that $f(e') = \{1\}$ and $f(e'') = \{2\}$. Therefore f is 2-Rainbow edge dominating function and W(f) = n. Thus γ'_{r2}(G) ≤ n (1)

The number of edges in $K_2 \square P_n$ is $q = 2(n-1) + n = 3n - 1$ and from the theorem

γ'_{r2} (G) ≥ $\left\lceil \frac{2q}{\Delta' + 2} \right\rceil$ and $\Delta'(G) = 4$ in $K_2 \square P_n$ we have

$$\gamma'_{r2}(G) \geq \left\lceil \frac{2(3n-1)}{6} \right\rceil = \left\lceil \frac{3n-1}{6} \right\rceil = n$$

Thus γ'_{r2}(G) ≥ n (2)

From (1) and (2) we have γ'_{r2}(G) = n. □

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