EUROPEAN ACADEMIC RESEARCH
Vol. IV, Issue 1/ April 2016

Impact Factor: 3.4546 (UIF)
DRJI Value: 5.9 (B+)

ISSN 2286-4822
www.euacademic.org

Fuzzy Translation and fuzzy multiplication of Q-algebras

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Abstract:

In this paper, we introduce the notion of fuzzy translation of Q-algebra, fuzzy extension ,fuzzy multiplication of Q-algebra and Cartesian product of fuzzy translation and fuzzy multiplication of fuzzy Q-ideal of Q-algebras and investigate some of their properties.

Key words: fuzzy Q-ideal, fuzzy translation**,** fuzzy extension, fuzzy multiplication, Cartesian product of Q-algebras.

2000 Mathematics Subject Classification: 06F35, 03G25, 08A72.

1. INTRODUCTION

I've been introducing a lot of concepts types algebra since 1966 and after studies have evolved and found Blur in 1965 by some researchers and these types Q-algebras where they were put up this concept in 2001, where he became the focus of our study in this research, and the introduction of the concept of fuzzy translation this kind of new idea for Q-algebras , and study some of the properties related to this type. The idea of fuzzy set it has been initiated by Zadeh L.A. in1965 [5]. In 2001, J. Neggers, S.S. and others [4] show an idea new notion, called Q-algebra. In [3] introducing the concept of Q-ideal and fuzzy Q-ideal in Q-algebra and study their characteristics, we study the fuzzy relations on Q-algebras.

In this paper we define a fuzzy translation and fuzzy multiplication of Q-algebras and look for some of their properties accurately by using the concepts of fuzzy Q-ideal and fuzzy Q-subalgebra We prove to that if μ and δ are fuzzy Q-ideals of Q-algebras X , then $\mu_{\gamma}^{M} \times \delta_{\alpha}^{M}$is a fuzzy Q-ideal of X×X. Also we show that if $\mu_{\gamma}^{M} \times \delta_{\alpha}^{M}$ is a fuzzy Q-ideal of X×X , either μ or δ is a fuzzy Q-ideal of X.

2. PRELIMINARIES

In this section we site the fundamental definitions that will be used in the sequel.

Definition 2.1 [6]:

Let(X;∗, 0) be a set with a binary operation (∗) and a constant (0).Then (X;∗, 0) is called a **Q-algebra** if it satisfies the following axioms: for all x, y ,z ∈X,

(1) $x * x = 0$,

(2) $x * 0 = x$,

(3) $(x * y) * z = (x * z) * y$.

For brevity we also call X a Q-algebra, We can define a binary relation (≤) by putting $x \le y$ if and only if $x * y = 0$,[4].

Definition 2.2 [4]:

Let (X;∗, 0) be a Q-algebra and S be a nonempty subset of X. Then S is called a **Q-subalgebra** of X if, $x * y$ ∈S, for any x, y ∈S.

Definition 2.3 [4]:

Let (X;∗, 0) be a Q-algebra and I be a nonempty subset of X. I is called a **Q-ideal** of X if it satisfies:

i. 0∈I,

ii. $(x * y) * z$ ∈I and y ∈ I imply $x * z \in$ I , for all x, y, z ∈X .

Definition 2.4 [7]:

Let X be a set. A fuzzy subset μ of X is a function μ: X →[0,1].

Definition 2.5 [4]:

Let X be a Q-algebra. A fuzzy subset μ of X is said to be a **fuzzy Q-subalgebra** of X if it satisfies: $\mu\,(x * y) \geq min\,\{\mu\,(x), \mu(y)\}$,for all x , y ∈ X.

Definition 2.6 [4]:

Let X be a Q-algebra. A fuzzy subset μ of X is said to be a **fuzzy Q-ideal** of X if it satisfies:

1. $\mu(0) \geq \mu(x)$, for all x∈ X,
2. $\mu(x * z) \geq min\{\mu((x * y) * z), \mu(y)\}$, for all x, y, z∈ X,

Proposition 2.7 [4]:

Let (X;∗, 0) by a Q-algebra, then every fuzzy Q-ideal of Q-algebra is a fuzzy Q-subalgebra.

3. FUZZY TRANSLATIONS OF FUZZY Q-IDEALS.

Now, we introduce the notion a fuzzy translation of Q-algebra as fuzzy translation of KUS-algebra and we study the relations among fuzzy translation, fuzzy extension of Q-ideal of Q-algebra X as in **[3,5].**

In what follows let (X; ∗ ,0) denote a Q-algebra, and for any fuzzy subset μ of X, we denote $T = 1 - \sup\{\mu(x) \mid x \in X\}$.

Definition 3.1 [1,5]:

Let X be a nonempty set and μ be a fuzzy subset of X and let $\alpha \in [0,T]$. A mapping $\mu_\alpha^T : X \to [0,1]$ is called a **fuzzy translation** of μ if it satisfies: $\mu_\alpha^T(x) = \mu(x) + \alpha$, for all $x \in X$.

Theorem 3.2 :

Let μ be a fuzzy subset of Q-algebra X and μ_α^T is fuzzy translation of μ for $\alpha \in [0,T]$. μ is a fuzzy Q-ideal of X if and only if μ_α^T is a fuzzy Q-ideal of X.

proof:

(⟹)

Assume μ be a fuzzy Q-ideal of X and let $\alpha \in [0,T]$. For all x, y, z ∈ X we have

1. since μ(0) ≥ μ(x) Then $\mu_\alpha^T(0) = \mu(0) + \alpha \geq \mu(x) + \alpha = \mu_\alpha^T(x)$.
2. $\mu_\alpha^T(x * z) = \mu(x * z) + \alpha \geq min\{\mu((x * y) * z), \mu(y)\} + \alpha$

$= min\{\mu((x * y) * z) + \alpha, \mu(y) + \alpha\} = min\{\mu_\alpha^T((x * y) * z), \mu_\alpha^T(y)\}$.

Hence μ_α^T is a fuzzy Q-ideal of X .

(⟸)

Assume the fuzzy translation μ_α^T is a fuzzy Q-ideal of X for some $\alpha \in [0,T]$.

Let x, y, z ∈ X, we have

1- $\mu(0) + \alpha = \mu_\alpha^T(0) \geq \mu_\alpha^T(x) = \mu(x) + \alpha \Rightarrow \mu(0) \geq \mu(x)$.

2- $\mu(x * z) + \alpha = \mu_\alpha^T(x * z) \geq min\{\mu_\alpha^T((x * y) * z), \mu_\alpha^T(y)\}$

$= min\{\mu((x * y) * z) + \alpha, \mu(y) + \alpha\} = min\{\mu((x * y) * z), \mu(y)\} + \alpha$

$\Rightarrow (x * z) \geq min\{\mu((x * y) * z), \mu(y)\}$. Hence μ is a fuzzy Q-ideal of X .⌂

Proposition 3.3 :

Let the fuzzy translation μ_α^T of μ be a fuzzy Q-ideal of X for $\alpha \in [0,T]$.If $x \leq y$ then $\mu_\alpha^T(x) \geq \mu_\alpha^T(y)$.

Proof:

Let x, y$\in$ X be such that $x \leq y$. Then $x * y = 0$ and hence $\mu_\alpha^T(x) = \mu_\alpha^T(x * 0)$
$\geq min\{\mu_\alpha^T((x * y) * 0), \mu_\alpha^T(y)\} = min\{\mu_\alpha^T(0), \mu_\alpha^T(y)\} = \mu_\alpha^T(y)$. ⌂

Definition3.4 [1]:

Let $\mu_1$and μ_2 be fuzzy subsets of a set X. If $\mu_1(x) \leq \mu_2(x)$ for all $x \in X$, then we say that μ_2 is **a fuzzy extension of** μ_1 .

Definition 3.5 :

Let μ_1 and $\mu_2$be fuzzy subsets of X. Then μ_2 is called **a fuzzy extension Q-ideal** of μ_1 if the following assertions are valid:
(I_1) μ_2 is a fuzzy extension of μ_1.
(I_2) If μ_1 is a fuzzy Q-ideal of X, then μ_2 is a fuzzy Q-ideal of X.

Proposition 3.6 :

Let μ be a fuzzy Q-ideal of X and let $\alpha, \gamma \in [0,T]$. If $\alpha \geq \gamma$, then the fuzzy translation μ_α^T of μ is a fuzzy extension Q-ideal of the fuzzy translation μ_γ^T of μ.

Proof:

Let μ be a fuzzy Q-ideal of X then by theorem (3.2) the fuzzy translation μ_γ^T of μ and the fuzzy translation μ_α^T of μ are fuzzy Q-ideals of X. for all $\alpha, \gamma \in [0,T]$ since $\alpha \geq \gamma$,
$\mu(x) + \alpha \geq \mu(x) + \gamma$, for all $x \in X$. Therefore $\mu_\alpha^T(x) \geq \mu_\gamma^T(x)$. Hence$\mu_\alpha^T$ is a fuzzy extension Q- ideal of μ_γ^T. ⌂

Proposition 3.7 :

For every fuzzy Q-ideal μ of X and $\gamma \in [0,T]$, the fuzzy translation μ_γ^T of μ is a fuzzy Q-ideal of X. If v is a fuzzy extension Q-ideal of μ_γ^{T}, then there exists α ∈ [0,T] such that $\alpha \geq \gamma$ and $\nu(x) \geq \mu_\alpha^{T}(x)$ for all x ∈ X.

Proof:

Let μ be a fuzzy Q-ideal of X and γ ∈ [0,T].then by theorem (3.2), μ_γ^{T} is a fuzzy Q-ideal of X . Let ν be a fuzzy extension Q-ideal of μ_γ^{T}.therefore $\nu(x) \geq \mu_\gamma^{T}(x)$ ∀ x ∈ X. Then choose $\alpha = \gamma + min\{\nu(x), \mu_\gamma^{T}(x)\}$. Clearly α ∈ [0,T] such that $\alpha \geq \gamma$. Then μ_α^{T} is a fuzzy translation μ and $\nu(x) \geq \mu_\alpha^{T}(x)$. Hence ν is also a fuzzy extension Q-ideal of the fuzzy μ_α^{T}.⌂

The following example illustrates proposition (3.7)

<u>Example 3.8</u>:

Let X ={0, 1,2} be a set with a binary operation (∗) defined by the following table:

∗	0	1	2
0	0	0	0
1	1	0	0
2	2	0	0

Then (X ;∗,0) is a Q-algebra by[4].Define a fuzzy subset μ of X by:

X	0	1	2
μ	0.8	0.4	0.4

Then μ is a fuzzy Q-ideal of X and T =1- 0.8= 0.2. If we take γ = 0.13, then the fuzzy Q-ideal translation μ_γ^{T} of μ is given by :

X	0	1	2
μ_γ^{T}	0.93	0.53	0.53

Let ν be a fuzzy subset of X defined by:

X	0	1	2
ν	0.98	0.57	0.57

then v is clearly a fuzzy extension Q-ideal of the fuzzy Q-ideal translation μ_{γ}^{T} of μ . But v is not a fuzzy Q-ideal translation μ_{α}^{T} of μ for all α ∈[0,T], since α in 0 ≠ α in 1 . Take α = 0.16, then α = 0.16 > 0.13 = γ and the fuzzy Q-ideal translation $\mu_{\alpha}^{T}$of μ is given as follows:

X	0	1	2
μ_{α}^{T}	0.96	0.56	0.56

Note that $\nu(x) \geq \mu_{\alpha}^{T}(x)$ for all x ∈ X, and hence v is a fuzzy extension Q-ideal of the fuzzy Q-ideal translation μ_{α}^{T} of μ.

Proposition 3.9 :

Let μ be a fuzzy Q-ideal of a Q-algebra X and α∈ [0,T]. Then the fuzzy translation μ_{α}^{T} of μ is a fuzzy extension Q-ideal of μ.

proof :

If μ is a fuzzy Q-ideal of X, then by theorem (3.2) ,the fuzzy translation μ_{α}^{T} of μ is also a fuzzy Q-ideal of X, for all α∈ [0,T]. Now $\mu_{\alpha}^{T}(x) = \mu(x) + \alpha \geq \mu(x)$,for all x ∈ X. Hence , the fuzzy translation μ_{α}^{T} is a fuzzy extension Q-ideal of μ. ⌂

A fuzzy extension Q-ideal of a fuzzy Q-ideal μ may not be represented as a fuzzy Q-ideal translation μ_{α}^{T} of μ , that is , the converse of proposition (3.9) is not true in general , as shown by the following example.

Example 3.10 :

Let X = {0,1,2,3} be a set with a binary operation (∗)defined by the following table:

∗	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	2
3	3	0	3	0

Then (X;∗,0) is a Q-algebra by [4]. Define a fuzzy subset μ of X by:

X	0	1	2	3
μ	0.8	0.4	0.4	0.5

Then μ is a fuzzy Q-ideal of X . Let v be a fuzzy subset of X defined by:

X	0	1	2	3
v	0.82	0.47	0.47	0.56

Then v is a fuzzy extension Q-ideal of μ . But it is not the fuzzy translation μ_α^T of μ for all α ∈ [0,T], since α in 0 ≠ α in 1.

Proposition 3.11 :

The intersection of any set of fuzzy Q-ideals translation of Q-algebra X is also fuzzy Q-ideal translation of X.

Proof:

Let $\{(\mu_\alpha^T)_i : i \in \Lambda\}$ be a family of fuzzy Q-ideals translation of Q-algebra X , then for any x, y, z ∈ X , i ∈Λ,

1- $\left(\bigcap_{i\in\Lambda}(\mu_\alpha^T)_i\right)(0) = \inf((\mu_\alpha^T)_i\,(0)) = \inf(\mu_i\,(0) + \alpha)$
$\geq \inf(\mu_i(x) + \alpha) \ = \inf((\mu_\alpha^T)_i\,(x)) = (\bigcap_{i\in\Lambda}\mu_\alpha^T)_i(x))$

2- $\left(\bigcap_{i\in\Lambda}(\mu_\alpha^T)_i\right)(x * z) \ = \inf((\mu_\alpha^T)_i\,(x * z)) = \ \inf(\mu_i(x * z) + \alpha)$
$\geq \inf\left(\min\{\mu_i\left((x * y) * z\right), \mu_i(y)\}\right) + \alpha \ = \ \inf\left(\min\{\mu_i\left((x * y) * z\right) + \alpha, \mu_i(y) + \alpha\}\right)$
$= \min\{\inf\left(\mu_i\left((x * y) * z\right) + \alpha\right), \inf(\mu_i\,(y) + \alpha)\}$
$= \min\{(\bigcap_{i\in\Lambda}\mu_i)\left((x * y) * z\right) + \alpha\,, (\bigcap_{i\in\Lambda}\mu_i)(y) + \alpha\}$
$= min\,\{\left(\bigcap_{i\in\Lambda}(\mu_\alpha^T)_i\right)((x * y) * z)\,, (\left(\bigcap_{i\in\Lambda}(\mu_\alpha^T)_i\right)(y)\}.$ ⌂

Proposition 3.12 :

The intersection of any set of fuzzy extension Q-ideals of a fuzzy Q-ideal μ of X is a fuzzy extension Q-ideal of μ.

Proof:

Let $\{\mu_i : i \in \Lambda\}$ be a family of fuzzy extension Q-ideals of a fuzzy Q-ideal μ of X ,Then

$\mu_i(x) \geq \mu(x)$ ∀ i ∈ Λ, x ∈ X since μ is a fuzzy Q-ideal of X. μ_i are fuzzy Q-ideals of X

∀ i ∈ Λ. Then $\cap_{i\in\Lambda} \mu_i$ is also a fuzzy ideal of X, by theorem(5.10) of [4].

Now $(\cap_{i\in\Lambda} \mu_i)(x) = inf_{i\in\Lambda}(\mu_i(x)) \geq inf\ (\mu(x)) = \mu(x)$. Hence $\cap_{i\in\Lambda} \mu_i$ is a fuzzy ideal extension of μ. ⌂

Clearly, the union of fuzzy extension Q-ideal of a fuzzy subset μ of X α is not a fuzzy extension Q-ideal of μ as seen in the following example.

Example 3.13 :

Let X = {0,1,2,3,4} be a set with binary operation(∗)defined by the following table:

∗	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	3	2	0

Then (X;∗,0) is a Q-algebra by[4].
Let μ , ν and δ be fuzzy subsets of X defined by:

X	0	1	2	3	4
μ	0.8	0.5	0.5	0.3	0.3
ν	0.9	0.8	0.6	0.8	0.6
δ	0.9	0.7	0.7	0.6	0.6

Respectively. Then ν and δ are fuzzy extension Q-ideals of μ obviously, the union ν ∪ δ is a fuzzy extension of μ, but it is not a fuzzy extension Q-ideals of μ since $(\nu \cup \delta)(4*0) = (\nu \cup \delta)(4) = 0.6 < 0.7 = \min\{(\nu \cup \delta)((4*2)*0), (\nu \cup \delta)(2)\} = \min\{(\nu \cup \delta)(3), (\nu \cup \delta)(2)\} = \min\{0.8, 0.7\}$.

Definition 3.14 [1]:

For a fuzzy subset μ of X, α∈ [0,T] and t ∈ [0,1] with $t \geq \alpha$,Let $U_\alpha(\mu;\, t) = \{x \in X \,:\, \mu(x) \geq t - \alpha\}$.

Theorem 3.15 :

let μ be a fuzzy subset of a Q-algebra X and α∈[0,T]. Then μ_α^T is a fuzzy Q-ideal translation of X if and only if $U_\alpha(\mu;\, t)$ is a Q-ideal of X ∀ t∈Im (μ) with t ≥ α.

Proof:

(⟹)

Assume that μ_α^T is a fuzzy Q-ideal translation of X and let t ∈ Im(μ) be such that t > α.

(1) Since $\mu_\alpha^T\ (0) \geq \mu_\alpha^T\ (x)$ for all x ∈ X, we have $\mu(0) + \alpha = \mu_\alpha^T\ (0) \geq \mu_\alpha^T\ (x)$ $= \mu(x) + \alpha$ that mean $\mu(0) \geq \mu(x)$, for all x∈ X. Let $x \in U_\alpha(\mu;\, t)$, then $\mu(x) \geq t - \alpha$ and $\mu(0) \geq \mu(x)$ imply $\mu(0) \geq \mu(x) \geq t - \alpha$. Hence $0 \in U_\alpha(\mu;\, t)$.

(2) Let x, y, z ∈ X be such that $(x * y) * z \in U_\alpha(\mu;\, t)$ and $y \in U_\alpha\ (\mu;\, t)$. Then $\mu((x * y) * z) \geq t - \alpha$ and $\mu(y) \geq t - \alpha$, i.e., $\mu_\alpha^T\ ((x * y) * z) = \mu((x * y) * z) + \alpha \geq t$ and $\mu_\alpha^T\ (y) = \mu(y) + \alpha \geq t$. Since $\mu_\alpha^T$is a fuzzy Q-ideal translation of X, it follows that

$\mu(x * z) + \alpha = \mu_\alpha^T(x * z) \geq min\{\mu_\alpha^T\ ((x * y) * z), \mu_\alpha^T(y)\} \geq t, that\ is,$

$\mu(x * z) \geq t - \alpha\ so\ that\ (x * z) \in U_\alpha\ (\mu;\, t)$. Therefore $U_\alpha(\mu;\, t)$ is Q-ideal of X.

(⟸)

suppose that $U_\alpha\ (\mu;\, t)$ is Q-ideal of X. for every t ∈ Im(μ) with t > α.

(1) If there exists x ∈ X such that $\mu_\alpha^T\ (0) < t \leq \mu_\alpha^T(x)$, then $\mu(x) \geq t - \alpha$ but $\mu(0) < t - \alpha$. This shows that $x \in U_\alpha\ (\mu;\, t)$ and $0 \notin U_\alpha\ (\mu;\, t)$. This is a contradiction, and so $\mu_\alpha^T(0) \geq \mu_\alpha^T\ (x)$, for all x ∈ X.

(2) Now assume that there exist x, y, z ∈ X such that

$\mu_\alpha^T\ (x * z) < \gamma \leq min\{\ \mu_\alpha^T\ ((x * y) * z), \mu_\alpha^T\ (y)\}$. Then $\mu((x * y) * z) \geq \gamma - \alpha$ and $\mu(y) \geq \gamma - \alpha$, but $\mu(x * z) < \gamma - \alpha$. Hence $((x * y) * z) \in U_\alpha(\mu;\ \gamma)$ and $y \in U_\alpha\ (\mu;\ \gamma)$, but $(x * z) \notin U_\alpha\ (\mu;\ \gamma)$, this is a contradiction. Therefore
$\mu_\alpha^T\ (x * z) \geq min\{\mu_\alpha^T\ ((x * y) * z),\ \mu_\alpha^T(y)\}$, for all x ,y, z ∈ X. Hence μ_α^T is a fuzzy Q-ideal translation of X. ⌂

Corollary 3.16 :

let μ be a fuzzy subset of a Q-algebra X and α∈[0,T]. Then μ is a fuzzy Q-ideal of X if and only if $U_\alpha(\mu;\ t)$ is a Q-ideal of X ∀ t∈Im(μ) with t ≥ α.

Proof:

By theorem(3.2) and theorem(3.15) .⌂

Proposition 3.17 :

Let μ be a fuzzy Q-ideal of a Q-algebra X and let α ∈ [0,T], then the fuzzy translation μ_α^T of μ is a fuzzy Q-subalgebra of X.

Proof:

Since μ be a fuzzy Q-ideal of a Q-algebra X , then by proposition (2.7) μ be a fuzzy Q-subalgebra of a Q-algebra X and let α ∈ [0,T] and x, y∈ X. Then $\mu_\alpha^T(x * y)$
$= \mu(x * y) + \alpha \geq min\{\mu(x), \mu(y)\} + \alpha = min\{\mu(x) + \alpha, \mu(y) + \alpha\}$
$= min\{\mu_\alpha^T\ (x),\ \mu_\alpha^T\ (y)\}$. Hence μ_α^T is a fuzzy Q-subalgebra translation of X. ⌂

Corollary 3.18 :

Let μ be a fuzzy Q-subalgebr of a Q-algebra X and let α ∈ [0,T], then the fuzzy translation μ_α^T of μ is a fuzzy Q-subalgebra of X.

In general, the converse of the proposition (3.17) is not true .

Example 3.19:

Let X = {0,1,2,3}be a Q-algebra which is given in Example (3.10). Define a fuzzy subset μ of X by:

X	0	1	2	3
μ	0.8	0.4	0.5	0.6

Then μ is not fuzzy Q-ideal of X. since μ(1∗3)= μ(1)=0.4<0.5=min{μ((1∗2)∗3),μ(2)}
=min {μ (0),μ (2)}, and T=0.2 .But if we take α=0.1 the fuzzy

X	0	1	2	3
μ_α^T	0.9	0.5	0.6	0.7

translation μ_α^T of μ is given as follows:

Then μ_α^T is a fuzzy Q-subalgera of X.

Proposition 3.20:

If the fuzzy translation μ_α^T of μ is a fuzzy Q-ideal of X, α ∈[0,T] then μ is a fuzzy Q-subalgebra.

Proof:

Since μ_α^T be a fuzzy Q-ideal of a Q-algebra X , then by proposition (2.7) μ_α^T be a fuzzy Q-subalgebra of a Q-algebra X and let α ∈ [0,T] and x, y ∈ X. Then $\mu(x * y) + \alpha$
$= \mu_\alpha^T(x * y) \geq min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\} = min\{\mu(x) + \alpha, \mu(y) + \alpha\}$
$= min\{\mu(x), \mu(y)\} + \alpha.$
Hence μ is a fuzzy Q-subalgebra of X. ⌂

Corollary 3.21 :

If the fuzzy translation μ_α^T of μ is a fuzzy Q-subalgebra of X, α ∈[0,T] then μ is a fuzzy Q-subalgebra.
In general, the converse of the proposition (3.20) is not true .

Example 3.22 :

Let X = {0, 1,2,3,4}be a Q-algebra which is given in Example (3.13). Define a fuzzy subset μ of X by:

X	0	1	2	3	4
μ	0.8	0.7	0.6	0.5	0.4

Then μ is a fuzzy Q-subalgera of X, and T=0.2 .But if we take α=0.02 the fuzzy translation $\mu_\alpha^T$of μ is given as follows:

X	0	1	2	3	4
μ_α^T	0.82	0.72	0.62	0.52	0.42

Then μ_α^T is not a fuzzy Q-ideal of X. since μ_α^T (4∗0) =μ_α^T (4)=0.42< 0.52=min{μ_α^T ((4∗2) ∗0), μ_α^T (2)}=min{μ_α^T (3), μ_α^T (2)}.

Definition 3.23[6]:

Let X and Y be Q-algebras. A mapping f : (X;∗ ,0) ⟶ (Y;$\acute{*}$, $\acute{0}$) is called a homomorphism if $f\ (x\ *\ y\ )=\ f\ (x\ )\ \acute{*}\ f\ (y\ )$, x , y ∈X.

Definition 3. 24[2]:

Let f : (X;∗ ,0) ⟶ (Y;$\acute{*}$, $\acute{0}$) be a mapping nonempty sets X and Y respectively . If μ is a fuzzy subset of X ,then

$$f(\mu)(y)=\begin{cases}\sup\limits_{x\in f^{-1}(y)}\mu(x), & f^{-1}(y)=\{x\in X|f(x)=y\}\neq\Phi\\ 0 & \text{otherwise}\end{cases}$$

is said to be the image of μ under f.

Proposition 3.25:

Let f : (X;∗ ,0) ⟶ (Y;$\acute{*}$, $\acute{0}$) be a homomorphism of Q-algebra X . For every the fuzzy translation μ_γ^T is a fuzzy Q-ideal of μ in X and α∈[0,T] ,then the image of μ_γ^T is also a fuzzy Q-ideal of μ in Y

Proof: Let $f : (X;*,0) \longrightarrow (Y;\acute{*},\acute{0})$ be a homomorphism between two Q-algebra X ,Y and the translation $\mu_{\gamma}^{\mathrm{T}}$ is a fuzzy Q-ideal of X then by theorem (4.5) of [4], $f\left(\mu_{\gamma}^{\mathrm{T}}\right)$ is a fuzzy Q-ideal of Y.⌂

Proposition 3.26:

Let f:X → Y be a homorphism of Q-algebra X in to a Q-algebra Y and $\mu_{\alpha}^{\mathrm{T}}$ be a fuzzy translation of μ ,then the pre-image of $\mu_{\alpha}^{\mathrm{T}}$ denoted by $f^{-1}(\mu_{\alpha}^{\mathrm{T}})$ is defined as $\{f^{-1}(\mu_{\alpha}^{\mathrm{T}})\} = \mu_{\alpha}^{\mathrm{T}}(f(x))$ ∀ x∈X. If μ is a fuzzy Q-ideal of Y, then $f^{-1}(\mu_{\alpha}^{\mathrm{T}})$ is a fuzzy Q-ideal of X.

Proof :

Let μ be a fuzzy Q-ideal of Y. Let x, y ,z ∈X.

(1) $f^{-1}\left(\mu_{\alpha}^{\mathrm{T}}(0)\right) = \mu_{\alpha}^{\mathrm{T}}\left(f(0)\right) = \mu\left(f(0)\right) + \alpha \geq \mu\left(f(x)\right) + \alpha = \mu_{\alpha}^{\mathrm{T}}\left(f(x)\right)$

$= f^{-1}(\mu_{\alpha}^{\mathrm{T}}(x)) \Longrightarrow f^{-1}(\mu_{\alpha}^{\mathrm{T}}(0)) \geq f^{-1}(\mu_{\alpha}^{\mathrm{T}}(0))$.

(2) $f^{-1}\left(\mu_{\alpha}^{\mathrm{T}}(x*z)\right) = \mu_{\alpha}^{\mathrm{T}}\left(f(x*z)\right) = \mu\left(f(x*z)\right) + \alpha$

$\geq min\{\mu(f((x*y)*z)), \mu(f(y)\} + \alpha$

$= min\{\mu(f((x*y)*z)) + \alpha, \mu(f(y) + \alpha\}$

$= min\{\mu_{\alpha}^{\mathrm{T}}(f((x*y)*z)), \mu_{\alpha}^{\mathrm{T}}(f(y)\} = min\{f^{-1}(\mu_{\alpha}^{\mathrm{T}}((x*y)*z)), f^{-1}(\mu_{\alpha}^{\mathrm{T}}(y)\} \Longrightarrow f^{-1}(\mu_{\alpha}^{\mathrm{T}}(x*z)) \geq min\{f^{-1}(\mu_{\alpha}^{\mathrm{T}}((x*y)*z)), f^{-1}(\mu_{\alpha}^{\mathrm{T}}(y)\}$.

Hence $f^{-1}(\mu_{\alpha}^{\mathrm{T}})$ is a fuzzy Q-ideal of X. ⌂

4-FUZZY MULTIPLICATIONS OF FUZZY Q-IDEALS:

We study the notion of fuzzy multiplication of Q-ideal of Q-algebra X and we give some properties of it as in [7] .

Definition 4.1[7]:

Let μ be a fuzzy subset of X and β∈ [0, 1].**A fuzzy multiplication** of μ, denoted by μ_{β}^{M} is defined to be a mapping μ_{β}^{M}: X→[0,1] define by $\mu_{\beta}^{\mathrm{M}}(x) = \beta \,.\, \mu(x)$, for all x ∈ X.

Theorem 4.2 :

Let μ be a fuzzy subset of a Q-algebra X and β ∈ (0,1].Then μ is a fuzzy Q-ideal of X if and only if the fuzzy multiplication μ_β^M is fuzzy Q-ideal of X .

Proof:

(⟹)

Assume μ is a fuzzy Q-ideal of X and let β ∈ (0, 1] .Then

(1) μ_β^M (0)= β. μ(0) ≥ β. μ(x) = μ_β^M (x)

(2) $\mu_\beta^M(x * z) = \beta.\mu(x * z) \geq \beta.min\{\mu((x * y) * z), \mu(y)\}$
$= min\{\beta.\mu((x * y) * z), \beta.\mu(y)\} = min\{\mu_\beta^M((x * y) * z), \mu_\beta^M(y)\}$ for all x, y, z ∈ X. Hence μ_β^M is a fuzzy Q-ideal of X. ⌂

(⟹)

Let β ∈ (0, 1] be such that μ_β^M is a fuzzy Q-ideal multiplication of X. Then for all x, y , z ∈ X

(1) $\beta.\mu(0) = \mu_\beta^M(0) \geq \mu_\beta^M(x) = \beta.\mu(x) \Longrightarrow \mu(0) \geq \mu(x)$

(2) $\beta.\mu(x * z) = \mu_\beta^M(x * z) \geq min\{\mu_\beta^M((x * y) * z), \mu_\beta^M(y)\}$
$= min\{\beta.\mu((x * y) * z), \beta.\mu(y)\} = \beta.min\{\mu((x * y) * z), \mu(y)\}$
$\Longrightarrow \mu(x * z) \geq min\{\mu((x * y) * z), \mu(y)\}$. Hence μ is a fuzzy Q-ideal of X. ⌂

Proposition 4.3 :

Let μ be a fuzzy subset of a Q-algebra X, α ∈ [0,T] and β ∈ (0, 1]. Then every fuzzy Q-ideal translation μ_α^T of μ is a fuzzy extension Q-ideal of the fuzzy Q-ideal multiplication μ_β^M of μ.

Proof:

For every x ∈ X **,**we have

(1) Assume that μ_β^M is a fuzzy Q-ideal of X. Then μ is a fuzzy Q-ideal of X by theorem (4.2). It follows from Theorem (3.2) that μ_α^T is a fuzzy Q-ideal of X for all α ∈ [0,T] .

(2) $\mu_{\alpha}^{\mathrm{T}}(x) = \mu(x) + \alpha \geq \mu(x) \geq \beta.\mu(x) = \mu_{\beta}^{\mathrm{M}}(x) \Rightarrow \mu_{\alpha}^{\mathrm{T}}$ is a fuzzy extension of μ_{β}^{M}. Hence every fuzzy translation $\mu_{\alpha}^{\mathrm{T}}$ is a fuzzy extension Q-ideal of the fuzzy Q-ideal multiplication μ_{β}^{M}.

⌂

The following example illustrates proposition (4.3)

Example 4.4 :

Consider a Q-algebra X = {0, 1, 2, 3,4} with the example (3.13). Define a fuzzy subset μ of X by :

X	0	1	2	3	4
μ	0.8	0.5	0.5	0.3	0.3

Then μ is a fuzzy Q-ideal of X. If we take ß= 0.2 then the fuzzy multiplication $\mu_{0.2}^{\mathrm{M}}$0f μ is given by :

X	0	1	2	3	4
$\mu_{0.2}^{\mathrm{M}}$	0.16	0.1	0.1	0.06	0.06

Clearly $\mu_{0.2}^{\mathrm{M}}$ is a fuzzy Q-ideal multiplication of X. Also, for any α ∈ [0, 0.2], the fuzzy translation $\mu_{\alpha}^{\mathrm{T}}$ of μ is given by:

X	0	1	2	3	4
$\mu_{\alpha}^{\mathrm{T}}$	0.8+ α	0.5+ α	0.5+ α	0.3+ α	0.3 +α

Then $\mu_{\alpha}^{\mathrm{T}}$ is a fuzzy extension of $\mu_{0.2}^{\mathrm{M}}$ and $\mu_{\alpha}^{\mathrm{T}}$is always a fuzzy Q-ideal translation of X for all α ∈ [0, 0.2]. Hence $\mu_{\alpha}^{\mathrm{T}}$ is a fuzzy extension Q-ideal of $\mu_{0.2}^{\mathrm{M}}$ for all α ∈ [0, 0.2].

Proposition 4.5 :

Let μ be a fuzzy Q-ideal of X and γ∈ [0, 1]. Then The fuzzy multiplication $\mu_{\gamma}^{\mathrm{M}}$of μ is a fuzzy Q-subalgebra of X.

Proof:

since μ fuzzy Q-ideal of a Q-algebra X, then by Proposition (2.7) μ be a fuzzy Q-subalgebra of Q-algebra let γ∈ (0, 1] and x, y ∈X. Then $\mu_{\gamma}^{\mathrm{M}}(x * y) = \gamma\,.\mu(x * y)$

$\geq \gamma . min\{ \mu (x), \mu (y)\} = min\{ \gamma . \mu (x), \gamma . \mu (y)\}$
$= min\{\mu_\gamma^{M} (x), \mu_\gamma^{M} (y)\}$.

Hence μ_γ^{M} of μ is a fuzzy Q-subalgebra multiplication of X. ⌂

In general, the converse of the proposition (4.5) is not true .

Example 4.6 :

Let X={0,1,2,3}be a Q-algebra which is given in Example(3.10).Define a fuzzy subset μ of X by :

X	0	1	2	3
μ	0.8	0.5	0.6	0.7

Then μ is not fuzzy Q-ideal of X. Since μ(1∗3)= μ(1)=0.5 < 0.6=min{ μ ((1∗2)∗3), μ (2)} =min{ μ (3),μ (2)} But if we take γ=0.2 the fuzzy multiplication μ_γ^{M} of μ is given as follows:

X	0	1	2	3
μ_γ^{M}	0.16	0.1	0.12	0.14

Then μ_γ^{M} is a fuzzy Q-subalgebra of X.

Proposition 4.7 :

If the fuzzy multiplication μ_γ^{M} of μ is a fuzzy Q-ideal of X, γ∈ (0,1] then μ is a fuzzy Q-subalgebra.

Proof:

Since μ_γ^{M} be a fuzzy Q-ideal of a Q-algebra X , then by proposition (2.7) μ_γ^{M} be a fuzzy Q-subalgebra of a Q-algebra X let γ∈ (0,1] and x, y ∈ X. Then $\gamma . \mu(x * y) = \mu_\gamma^{M}(x * y)$
$\geq min\{\mu_\gamma^{M}(x), \mu_\gamma^{M}(y)\} = min\{\gamma . \mu(x), \gamma . \mu(y)\} =$
$\gamma . min\{\mu(x), \mu(y)\}$since γ ≠ 0 it follows that $\mu(x * y) \geq min\{\mu(x), \mu(y)\}$. Hence μ is a fuzzy Q-subalgebra of X. ⌂

Corollary 4.8 :

If the fuzzy multiplication μ_γ^M of μ is a fuzzy Q-subalgebra of X, γ∈ (0,1] then μ is a fuzzy Q-subalgebra.

In general, the converse of the proposition (4.7) is not true.

Example 4.9 :

Let X={0,1,2,3,4}be a Q-algebra which is given in Example(3.13).Define a fuzzy subset μ of X by :

X	0	1	2	3	4
μ	0.8	0.6	0.5	0.4	0.3

Then μ is a fuzzy Q-subalgera of X. But if we take γ=0.03 the fuzzy multiplication μ_γ^M of μ is given as follows:

X	0	1	2	3	4
μ_γ^M	0.024	0.018	0.015	0.012	0.009

Then μ_γ^M is not a fuzzy Q-ideal multiplication of X. sinceμ_γ^M (4∗0)= μ_γ^M (4) = 0.009 < 0.012 =min{μ_γ^M ((4∗2)∗ 0), μ_γ^M (2)}=min{μ_γ^M (3), μ_γ^M (2)} .

Proposition 4.10:

If the fuzzy multiplication μ_γ^M of μ is a fuzzy Q-ideal of X, α ∈[0,T] then μ is a fuzzy Q-subalgebra.

Proof:

Since μ_γ^M be a fuzzy Q-ideal of a Q-algebra X , then by proposition (2.7) μ_α^T be a fuzzy Q-subalgebra of a Q-algebra X and let α ∈ [0,T] and x, y ∈ X. Then $\beta.\mu(x*y) = \mu_\gamma^M(x*y) \geq min\{\mu_\gamma^M(x), \mu_\gamma^M(y)\}$
$= min\{\beta.\mu(x), \beta.\mu(y)\} = \beta.min\{\mu(x), \mu(y)\}$. Hence μ is a fuzzy Q-subalgebra of X. ⌂

Corollary 4.11 :

If the fuzzy multiplication μ_{γ}^{M} of μ is a fuzzy Q-subalgebra of X, α ∈[0,T] then μ is a fuzzy Q-subalgebra.

In general, the converse of the proposition (4.10) is not true .

Example 4.12 :

Let X = {0, 1,2,3,4}be a Q-algebra which is given in Example (3.13). Define a fuzzy subset μ of X by:

X	0	1	2	3	4
μ	0.8	0.7	0.6	0.5	0.4

Then μ is a fuzzy Q-subalgera of X, and T=0.2 .But if we take β=0.02 the fuzzy multiplication $\mu_{\gamma}^{M}$of μ is given as follows:

X	0	1	2	3	4
μ_{γ}^{M}	0.16	0.14	0.12	0.10	0.08

Then μ_{γ}^{M} is not a fuzzy Q-ideal of X. sinceμ_{γ}^{M} (4∗0) =μ_{γ}^{M} (4)=0.08< 0.10=min{μ_{γ}^{M} ((4∗2) ∗0), μ_{γ}^{M} (2)}=min{μ_{γ}^{M} (3), μ_{γ}^{M} (2)}.

Proposition 4.13:

Let $f:(X;*,0)\longrightarrow(Y;\acute{*},\acute{0})$be a homomorphism of Q-algebra X . For every the fuzzy multiplication μ_{γ}^{M} is a fuzzy Q-ideal of μ in X and α∈[0,T] ,then the image of μ_{γ}^{M} is also a fuzzy Q-ideal of μ in Y.

Proof :

Let $f:(X;*,0)\longrightarrow(Y;\acute{*},\acute{0})$be a homomorphism between two Q-algebra X ,Y and the multiplication μ_{γ}^{M} is a fuzzy Q-ideal of X then by theorem (4.5) of [3], $f\left(\mu_{\gamma}^{M}\right)$ is a fuzzy Q-ideal of Y.⌂

Proposition 4.14:

Let $f:(X;*,0)\longrightarrow(Y;\acute{*},\acute{0})$be a homomorphism of Q-algebra X in to a Q-algebra Y and μ_{γ}^{M} be a fuzzy multiplication of μ ,then the pre-image of μ_γ^{M} denoted by $f^{-1}\left(\mu_\gamma^{\mathrm{M}}\right)$ is defined as $\{f^{-1}\left(\mu_\gamma^{\mathrm{M}}\right)\} = \mu_\gamma^{\mathrm{M}}\ (f\ (x))$ ∀ x∈X. If µ is a fuzzy Q-ideal of Y, then $f^{-1}\left(\mu_\gamma^{\mathrm{M}}\right)$ is a fuzzy Q-ideal of X.

Proof :

Let µ be a fuzzy Q-ideal of Y. Let x, y ,z ∈X.

(1) $f^{-1}(\mu_\gamma^{\mathrm{M}}\ (0)) = \mu_\gamma^{\mathrm{M}}\left(f\ (0)\right) = \gamma.\mu\left(f\ (0)\right) \geq \gamma.\mu\left(f\ (x)\right) =$

$\mu_\gamma^{\mathrm{M}}\left(f\ (x)\right) = f^{-1}\left(\mu_\gamma^{M}\ (x)\right) \Rightarrow f^{-1}(\mu_\gamma^{M}\ (0)) \geq f^{-1}(\mu_\gamma^{M}\ (0))$.

(2) $f^{-1}\left(\mu_\gamma^{\mathrm{M}}\ (x*z)\right) = \mu_\gamma^{M}\left(f\ (x*z)\right) = \gamma.\mu\left(f\ (x*z)\right)$

$\geq \gamma.min\{\ \mu(f\ ((x*y)*z)), \mu(f\ (y)\}$

$= min\{\ \gamma.\mu(f((x*y)*z)), \gamma.\mu(f(y)\}$

$= min\{\mu_\gamma^{M}(f((x*y)*z)), \mu_\gamma^{M}(f\ (y)\}$

$= min\{f^{-1}(\mu_\gamma^{\mathrm{M}}\ ((x*y)*z)), (f^{-1}(\mu_\gamma^{\mathrm{M}}\ (y)\}$

$\Rightarrow f^{-1}(\mu_\gamma^{\mathrm{M}}\ (x*z)) \geq min\{f^{-1}(\mu_\gamma^{\mathrm{M}}\ ((x*y)*z)), f^{-1}(\mu_\gamma^{\mathrm{M}}\ (y)\}$.Hence $f^{-1}(\mu_\gamma^{\mathrm{M}})$ is a fuzzy Q-ideal of X .⌂

5-CARTESIAN PRODUCT ON FUZZY TRANSLATION AND FUZZY MULTIPLICATION

In this section, we discuss the Cartesian product of fuzzy translation and fuzzy multiplication of Q-algebras and establish some of its properties in detail on the basis of fuzzy Q-ideal as [3].

Definition 5.1:

Let μ_α^{T} and $\delta_\alpha^{\mathrm{T}}$ be fuzzy translations of a Q-algebra X. The Cartesian product $\mu_\alpha^{\mathrm{T}} \times \delta_\alpha^{\mathrm{T}}$:X×X→[0,1] is defined by $(\mu_\alpha^{\mathrm{T}} \times \delta_\alpha^{\mathrm{T}})(x\ ,y) = min\{\mu_\alpha^{\mathrm{T}}(x), \delta_\alpha^{\mathrm{T}}(y)\}$,for all x, y ∈X.

Theorem 5.2:

Let µ and v be two fuzzy Q-ideals of a Q-algebra X. Let $\mathrm{T} = min\{\mathrm{T}_\mu, \mathrm{T}_v\}$where

$T_\mu = 1 - sup\{ \mu(x) : x \in X \}$ and $T_\nu = 1 - sup\{ \nu(x) : x \in X \}$ and $\alpha \in [0,T]$. Then the fuzzy translation of Cartesian product $\mu \times \nu$ is a fuzzy Q-ideal of X×X.

Proof:

Let μ and ν be two fuzzy Q-ideals of a Q-algebra X. let $\alpha \in [0,T]$.Now, by theorem (3.2) ,μ_α^T ,ν_α^T are fuzzy Q-ideals of X and by theorem (6.9) of [4],$\mu_\alpha^T \times \nu_\alpha^T$ is a fuzzy Q-ideal of X×X. Also $(\mu \times \nu)_\alpha^T(x, y) = (\mu \times \nu)(x, y) + \alpha = min\{ \mu(x), \nu(y)\} + \alpha$

$= min\{ \mu(x) + \alpha, \nu(y) + \alpha\} = min\{ \mu_\alpha^T(x), \nu_\alpha^T(y)\} = (\mu_\alpha^T \times \nu_\alpha^T)(x, y) \forall (x, y) \in X \times X$.

Hence $(\mu \times \nu)_\alpha^T$ is a fuzzy Q-ideal of X×X .⌂

Theorem 5.3:

Let μ and δ be fuzzy subsets in a Q-algebra X such that $\mu_\alpha^T \times \delta_\alpha^T$ is a fuzzy Q-ideal of X×X. Then :

(i) Either $\mu_\alpha^T(0) \geq \mu_\alpha^T(x)$ or $\delta_\alpha^T(0) \geq \delta_\alpha^T(x)$ for all $x \in X$

(ii) If $\mu_\alpha^T(0) \geq \mu_\alpha^T(x)$ for all $x \in X$. then either $\delta_\alpha^T(0) \geq \mu_\alpha^T(x)$ or $\delta_\alpha^T(0) \geq \delta_\alpha^T(x)$

(iii) If $\delta_\alpha^T(0) \geq \delta_\alpha^T(x)$ for all $x \in X$. Then either $\mu_\alpha^T(0) \geq \mu_\alpha^T(x)$ or $\mu_\alpha^T(0) \geq \delta_\alpha^T(x)$

Proof:

(i)→(ii) Let $\mu_\alpha^T \times \delta_\alpha^T$be a fuzzy Q-ideal of X×X.(i) suppose that $\mu_\alpha^T(0) < \mu_\alpha^T(x)$ and $\delta_\alpha^T(0) < \delta_\alpha^T(x)$for some x, y∈X .Then $(\mu_\alpha^T \times \delta_\alpha^T)(x, y) = min\{\mu_\alpha^T(x), \delta_\alpha^T(y)\} > \min\{\mu_\alpha^T(0), \delta_\alpha^T(0)\} = (\mu_\alpha^T \times \delta_\alpha^T)(0,0)$.which is a contradiction .Therefore $\mu_\alpha^T(0) \geq \mu_\alpha^T(x)$ or $\delta_\alpha^T(0) \geq \delta_\alpha^T(x)$ for all x∈X.

(ii) →(iii) Assume that there exists x, y ∈ X such that $\delta_\alpha^T(0) < \mu_\alpha^T(x)$ and $\delta_\alpha^T(0) < \delta_\alpha^T(x)$.Then $(\mu_\alpha^T \times \delta_\alpha^T)(0,0) = min\{\mu_\alpha^T(0), \delta_\alpha^T(0)\} = \delta_\alpha^T(0)$ *and hence*

$(\mu_\alpha^{\mathrm{T}} \times \delta_\alpha^{\mathrm{T}})(x,y) = min\{\mu_\alpha^{\mathrm{T}}(x), \delta_\alpha^{\mathrm{T}}(y)\} > \delta_\alpha^{\mathrm{T}}(0) = (\mu_\alpha^{\mathrm{T}} \times \delta_\alpha^{\mathrm{T}})(0,0)$ which is a contradiction. Hence, if $\mu_\alpha^{\mathrm{T}}(0) \geq \mu_\alpha^{\mathrm{T}}(x)$ for all x∈X. then either $\delta_\alpha^{\mathrm{T}}(0) \geq \mu_\alpha^{\mathrm{T}}(x)$ or $\delta_\alpha^{\mathrm{T}}(0) \geq \delta_\alpha^{\mathrm{T}}(x)$.

(iii)→(i) it is clear .⌂

Theorem 5.4:

Let μ and δ be fuzzy subsets of a Q-algebra X such that $\mu_\alpha^{\mathrm{T}} \times \delta_\alpha^{\mathrm{T}}$ is a fuzzy Q-ideal of X×X. Then either μ or δ is a fuzzy Q-ideal of X.

Proof:

First we prove that δ is a fuzzy Q-ideal of X. Since by theorem (5.3(i)) either

$\mu_\alpha^{\mathrm{T}}(0) \geq \mu_\alpha^{\mathrm{T}}(x)$ or $\delta_\alpha^{\mathrm{T}}(0) \geq \delta_\alpha^{\mathrm{T}}(x)$ for all x∈X .Assume that $\delta_\alpha^{\mathrm{T}}(0) \geq \delta_\alpha^{\mathrm{T}}(x)$ for all x∈X $\Longrightarrow \delta(0) + \alpha \geq \delta(x) + \alpha. \Longrightarrow \delta(0) \geq \delta(x)$.

It follows from theorem 5.3(iii) that either $\mu_\alpha^{\mathrm{T}}(0) \geq \mu_\alpha^{\mathrm{T}}(x)$ or $\mu_\alpha^{\mathrm{T}}(0) \geq \delta_\alpha^{\mathrm{T}}(x)$.

If $\mu_\alpha^{\mathrm{T}}(0) \geq \delta_\alpha^{\mathrm{T}}(x)$ for any x∈X $(\mu_\alpha^{\mathrm{T}} \times \delta_\alpha^{\mathrm{T}})(0,x) = min\{\mu_\alpha^{\mathrm{T}}(0), \delta_\alpha^{\mathrm{T}}(x)\} = \delta_\alpha^{\mathrm{T}}(x)$. Let (x_1,x_2), (y_1,y_2),(z_1,z_2) ∈X×X since $\mu_\alpha^{\mathrm{T}} \times \delta_\alpha^{\mathrm{T}}$ is a fuzzy Q-ideal of X×X. we have

$$(\mu_\gamma^{\mathrm{T}} \times \delta_\gamma^{\mathrm{T}})(x_1 * z_1, x_2 * z_2) = min\{(\mu_\gamma^{\mathrm{T}}(x_1 * z_1), \delta_\gamma^{\mathrm{T}}(x_2 * z_2)\}$$
$$\geq min\{min\{\mu_\alpha^{\mathrm{T}}((x_1 * y_1) * z_1), \mu_\alpha^{\mathrm{T}}(y_1)\}, min\{\delta_\alpha^{\mathrm{T}}((x_2 * y_2) * z_2), \delta_\alpha^{\mathrm{T}}(y_2)\}$$
$$= min\{min\{\mu_\alpha^{\mathrm{T}}((x_1 * y_1) * z_1), \delta_\alpha^{\mathrm{T}}((x_2 * y_2)z_2)\}, min\{\mu_\alpha^{\mathrm{T}}(y_1), \delta_\alpha^{\mathrm{T}}(y_2)\}$$
$$= min\{(\mu_\alpha^{\mathrm{T}} \times \delta_\alpha^{\mathrm{T}})(((x_1 * y_1) * z_1), ((x_2 * y_2) * z_2)), (\mu_\alpha^{\mathrm{T}} \times \delta_\alpha^{\mathrm{T}})(y_1, y_2)\} \text{ ---------- (A)}$$

If we take $x_1=y_1=z_1=0$ in (A), then $\delta(x_2 * z_2) + \alpha = \delta_\alpha^{\mathrm{T}}(x_2 * z_2)$

$$= (\mu_\alpha^{\mathrm{T}} \times \delta_\alpha^{\mathrm{T}})(0, x_2 * z_2)$$
$$\geq min\{(\mu_\alpha^{\mathrm{T}} \times \delta_\alpha^{\mathrm{T}})(0, ((x_2 * y_2) * z_2), (\mu_\alpha^{\mathrm{T}} \times \delta_\alpha^{\mathrm{T}})(0, y_2)\}$$
$$= \min[min\{(\mu_\alpha^{\mathrm{T}}(0), \delta_\alpha^{\mathrm{T}}((x_2 * y_2) * z_2)\}, min\{\mu_\alpha^{\mathrm{T}}(0), \delta_\alpha^{\mathrm{T}}(y_2)\}]$$

$= min\{\delta_\alpha^T((x_2 * y_2) * z_2), \delta_\alpha^T(y_2)\}$

$= min\{\delta((x_2 * y_2) * z_2) + \alpha, \delta(y_2) + \alpha\}$

$= min\{\delta((x_2 * y_2) * z_2), \delta(y_2)\} + \alpha$ this prove that δ is a fuzzy Q-ideal of X.

Next we will prove that μ is a fuzzy Q-ideal of X. Let $\mu_\alpha^T(0) \geq \mu_\alpha^T(x) \Longrightarrow \mu(0) \geq \mu(x)$ since by theorem 4.2(ii) either $\delta_\alpha^T(0) \geq \mu_\alpha^T(x)$ or $\delta_\alpha^T(0) \geq \delta_\alpha^T(x)$ If $\delta_\alpha^T(0) \geq \mu_\alpha^T(x)$for any x∈X. Hence $(\mu_\alpha^T \times \delta_\alpha^T)(x, 0) = \min\{\mu_\alpha^T(x), \delta_\alpha^T(0)\} = \mu_\alpha^T(x)$,taking $\mu_\alpha^T(x_1)$

$= (\mu_\alpha^T \times \delta_\alpha^T)(x_1, 0)$ If we take x_2=y_2=z_2=0 in (A),then $\mu(x_1 * z_1) + \alpha = \mu_\alpha^T(x_1 * z_1)$

$= (\mu_\alpha^T \times \delta_\alpha^T)(x_1 * z_1, 0)$

$\geq min\left\{(\mu_\alpha^T \times \delta_\alpha^T)\left(((x_1 * y_1) * z_1), 0\right), (\mu_\alpha^T \times \delta_\alpha^T)(y_1, 0)\right\}$

$= min[min\{(\mu_\alpha^T((x_1 * y_1) * z_1), \delta_\alpha^T(0)\}, min\{\mu_\alpha^T(y_1), \delta_\alpha^T(0)\}]$

$= \min\{\mu_\alpha^T((x_1 * y_1) * z_1), \delta_\alpha^T(y_1)\}$

$= \min\{\mu((x_1 * y_1) * z_1) + \alpha, \delta(y_1) + \alpha\}$

$= min\{\mu((x_1 * y_1) * z_1), \delta(y_1)\} + \alpha$ which proves that μ is a fuzzy Q-ideal of X.

Hence either μ or δ is a fuzzy Q- ideal of X.⌂

Theorem 5.5:

If μ and δ are fuzzy Q-ideals in a Q-algebra X, then $\mu_\gamma^M \times \delta_\alpha^M$is a fuzzy Q-ideal in X×X

Proof:

Let μ and δ be two fuzzy Q-ideals of a Q-algebra X. let γ∈(0,1]. Now by theorem (3.2) $\mu_\gamma^M, \delta_\alpha^M$ are fuzzy Q-ideal of X and by theorem (6.9) of [4] . $\mu_\gamma^M \times \delta_\alpha^M$ is a fuzzy Q-ideal of X×X. Also $(\mu \times \delta)_\gamma^M(x, y) = \gamma.(\mu \times \delta)(x, y) = \gamma. min\{\mu(x), \delta(y)\}$

$= min\{\gamma.\mu(x), \gamma.\delta(y)\} = min\{\mu_\gamma^M(x), \delta_\alpha^M(y)\} = (\mu_\gamma^M \times \delta_\alpha^M)(x, y)$,∀(x ,y) ∈X×X.

Hence $(\mu \times \delta)_\gamma^M$ is a fuzzy Q-ideal of X×X .⌂

Theorem 5.6:

Let μ and δ be fuzzy subsets in a Q-algebra X such that $\mu_{\gamma}^{M} \times \delta_{\alpha}^{M}$ is a fuzzy Q-ideal of X×X. Then :

(i) Either $\mu_{\gamma}^{M}(0) \geq \mu_{\gamma}^{M}(x)$ or $\delta_{\alpha}^{M}(0) \geq \delta_{\alpha}^{M}(x)$ for all x∈X.

(ii) If $\mu_{\gamma}^{M}(0) \geq \mu_{\gamma}^{M}(x)$ for all x∈X. then either $\delta_{\gamma}^{M}(0) \geq \mu_{\gamma}^{M}(x)$ or $\delta_{\gamma}^{M}(0) \geq \delta_{\gamma}^{M}(x)$.

(iii) If $\delta_{\gamma}^{M}(0) \geq \delta_{\gamma}^{M}(x)$ for all x∈X. Then either $\mu_{\gamma}^{M}(0) \geq \mu_{\gamma}^{M}(x)$ or $\mu_{\gamma}^{M}(0) \geq \delta_{\gamma}^{M}(x)$.

Proof:

Let $\mu_{\gamma}^{M} \times \delta_{\alpha}^{M}$ be a fuzzy Q-ideal of X×X.

(i) →(ii) suppose that $\mu_{\gamma}^{M}(0) < \mu_{\gamma}^{M}(x)$ and $\delta_{\alpha}^{M}(0) < \delta_{\alpha}^{M}(x)$ for some x ,y∈X. Then

$(\mu_{\gamma}^{M} \times \delta_{\alpha}^{M})(x,y) = min\{\mu_{\gamma}^{M}(x), \delta_{\alpha}^{M}(y)\} > min\{\mu_{\gamma}^{M}(0), \delta_{\alpha}^{M}(0)\} = (\mu_{\gamma}^{M} \times \delta_{\alpha}^{M})(0,0)$. which is a contradiction .Therefore $\mu_{\gamma}^{M}(0) \geq \mu_{\gamma}^{M}(x)$ or $\delta_{\alpha}^{M}(0) \geq \delta_{\alpha}^{M}(x)$ for all x∈X.

(ii)→(iii) Assume that there exists x, y∈ X such that $\delta_{\gamma}^{M}(0) < \mu_{\gamma}^{M}(x)$ and $\delta_{\gamma}^{M}(0) < \delta_{\gamma}^{M}(x)$.Then $(\mu_{\gamma}^{M} \times \delta_{\gamma}^{M})(0,0) = min\{\mu_{\gamma}^{M}(0), \delta_{\gamma}^{M}(0)\} = \delta_{\gamma}^{M}(0)$ and hence $(\mu_{\gamma}^{M} \times \delta_{\gamma}^{M})(x,y)$ $= min\{\mu_{\gamma}^{M}(x), \delta_{\gamma}^{M}(y)\} > \delta_{\gamma}^{M}(0) = (\mu_{\gamma}^{M} \times \delta_{\gamma}^{M})(0,0)$ which is a contradiction .Hence ,if $\mu_{\gamma}^{M}(0) \geq \mu_{\gamma}^{M}(x)$ for all x∈X. then either $\delta_{\gamma}^{M}(0) \geq \mu_{\gamma}^{M}(x)$ or $\delta_{\gamma}^{M}(0) \geq \delta_{\gamma}^{M}(x)$.

(iii)→(i) it is clear .⌂

Theorem 5.7:

Let μ and δ be fuzzy subsets of a Q-algebra X such that $\mu_{\gamma}^{M} \times \delta_{\alpha}^{M}$ is a fuzzy Q-ideal of X×X. Then either μ or δ is a fuzzy Q-ideal of X.

Proof:

First we prove that δ is a fuzzy Q-ideal of X. Since by theorem (5.6(i)) either

$\mu_\gamma^M(0) \geq \mu_\gamma^M(x)$ or $\mu_\gamma^M(0) \geq \delta_\gamma^M(x)$ for all x∈X .Assume that $\delta_\gamma^M(0) \geq \delta_\gamma^M(x)$ for all x∈X.$\Longrightarrow \gamma.\delta(0) \geq \gamma.\delta(x) \Longrightarrow \delta(0) \geq \delta(x)$. It follows from theorem 5.6(iii) that either $\mu_\gamma^M(0) \geq \mu_\gamma^M(x)$or $\mu_\gamma^M(0) \geq \delta_\gamma^M(x)$.

If $\mu_\gamma^M(0) \geq \delta_\gamma^M(x)$for any x∈X then$(\mu_\gamma^M \times \delta_\gamma^M)(0,x) = min\{\mu_\gamma^M(0), \delta_\gamma^M(x)\} = \delta_\gamma^M(x)$.

Let $(x_1,x_2),(y_1,y_2),(z_1,z_2)$ ∈X×X since $\mu_\gamma^M \times \delta_\gamma^M$ is a fuzzy Q-ideal of X×X. we have

$(\mu_\gamma^M \times \delta_\gamma^M)(x_1 * z_1, x_2 * z_2) = min\{(\mu_\gamma^M(x_1 * z_1), \delta_\gamma^M(x_2 * z_2)\}$

$\geq min\{min\{\mu_\gamma^M((x_1 * y_1) * z_1), \mu_\gamma^M(y_1)\}, min\{\delta_\gamma^M((x_2 * y_2) * z_2), \delta_\gamma^M(y_2)\}$

$= min\{min\{\mu_\gamma^M((x_1 * y_1) * z_1), \delta_\gamma^M((x_2 * y_2)z_2)\}, min\{\mu_\gamma^M(y_1), \delta_\gamma^M(y_2)\}$

$= min\{(\mu_\gamma^M \times \delta_\gamma^M)(((x_1 * y_1) * z_1), ((x_2 * y_2) * z_2)), (\mu_\gamma^M \times \delta_\gamma^M)(y_1, y_2)\}$ --------- (A)

If we take $x_1=y_1=z_1=0$ in (A), then $\delta(x_2 * z_2) + \alpha = \delta_\gamma^M(x_2 * z_2)$

$= (\mu_\gamma^M \times \delta_\gamma^M)(0, x_2 * z_2)$

$\geq min\{(\mu_\gamma^M \times \delta_\gamma^M)(0, ((x_2 * y_2) * z_2), (\mu_\gamma^M \times \delta_\gamma^M)(0, y_2)\}$

$= \min[min\{(\mu_\gamma^M(0), \delta_\gamma^M((x_2 * y_2) * z_2)\}, min\{\mu_\gamma^M(0), \delta_\gamma^M(y_2)\}]$

$= min\{\delta_\gamma^M((x_2 * y_2) * z_2), \delta_\gamma^M(y_2)\}$

$= min\{\gamma.\delta((x_2 * y_2) * z_2), \gamma.\delta(y_2)\}$

$= \gamma.min\{\delta((x_2 * y_2) * z_2), \delta(y_2)\}$ this prove that δ is a fuzzy Q-ideal of X.

Next we will prove that μ is a fuzzy Q-ideal of X. Let $\mu_\gamma^M(0) \geq \mu_\gamma^M(x)$.$\Longrightarrow \mu(0) \geq \mu(x)$ since by theorem 4.2(ii) either$\delta_\gamma^M(0) \geq \mu_\gamma^M(x)$or $\delta_\gamma^M(0) \geq \delta_\gamma^M(x)$.If $\delta_\gamma^M(0) \geq \mu_\gamma^M(x)$for any x∈X .Hence$(\mu_\gamma^M \times \delta_\gamma^M)(x,0) = min\{\mu_\gamma^M(x), \delta_\gamma^M(0)\} = \mu_\gamma^M(x)$taking $\mu_\gamma^M(x_1) = (\mu_\gamma^M \times \delta_\gamma^M)(x_1, 0)$ If we take $x_2=y_2=z_2=0$ in (A),then $\gamma.\mu(x_1 * z_1) = \mu_\gamma^M(x_1 * z_1)$

$= (\mu_\gamma^M \times \delta_\gamma^M)(x_1 * z_1, 0)$

$\geq min\{(\mu_\gamma^M \times \delta_\gamma^M)(((x_1 * y_1) * z_1), 0), (\mu_\gamma^M \times \delta_\gamma^M)(y_1, 0)\}$

$= min[min\{(\mu_\gamma^M((x_1 * y_1) * z_1), \delta_\gamma^M(0)\}, min\{\mu_\gamma^M(y_1), \delta_\gamma^M(0)\}]$

$= \min\left\{\mu_\gamma^M((x_1 * y_1) * z_1), \delta_\alpha^T(y_1)\right\}$

$= \min\{\gamma.\mu((x_1 * y_1) * z_1), \gamma.\delta(y_1)\}$

$= \gamma.min\{\mu((x_1 * y_1) * z_1), \delta(y_1)\}$ which proves that μ is a fuzzy Q-ideal of X. Hence either μ or δ is a fuzzy Q- ideal of X.⌂

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