EUROPEAN ACADEMIC RESEARCH
Vol. IV, Issue 2/ May 2016

Impact Factor: 3.4546 (UIF)
DRJI Value: 5.9 (B+)

ISSN 2286-4822
www.euacademic.org

Some fixed point results in ultrametric spaces

Dr. SANJAY KUMAR TIWARI
Associate Professor
P. G. Department of Mathematics
M.U.Bodh Gaya, India
CHANCHLESH KUMAR
Research Scholar
P. G. Department of Mathematics
M.U.Bodh Gaya, India

Abstract:

In the present paper, we obtain some new fixed point theorems for relative contractive mappings in the setting of ultrametric spaces. Our theorems complement, generalize and extend some well known results of Gajic [7], Rao and Kishore[5] and others.

Key words: fixed point theorems, ultrametric spaces

1. INTRODUCTION:

There have been a lot of generalizations of metric space such as cone metric space, G- metric space, b- metric space, probabilistic metric space, Ultrametric space etc. One of the recent generalization is ultrametric space defined by Rooji [1]. After the introduction of ultrametric space, Petals and Vidalis [2] proved a fixed point theorem for contractive mappings on spherically complete ultrametric space X.

Petals and Vidalis [2] established the following fixed point theorem:

Theorem (1.1): Let (X,d) be a spherically complete ultrametric space and $T:X \to X$ a contractive mapping. Then T has a unique fixed point.

In 2001 Gajic [7] obtained the following generalization of the above theorem:

Theorem (1.2): Let (X,d) be a spherically complete ultrametric space and $T:X \to X$ a mapping such that for all $x,y \in X$, $x \neq y$,

$$d(Tx,Ty) < \max \{ d(x,y), d(x,Tx), d(y,Ty)\}$$

Then T has a unique fixed point.

Later on Rao and Kishore [5] extended the above result for a pair of maps of Jungck type as follows:

Theorem (1.3): Let (X,d) be a spherically complete ultrametric space. If f and T are self maps on X satisfying

$T(X) \subseteq f(X)$

$d(Tx,Ty)<\max\{d(f(f(x),f(y)), d(f(x),T(x)),d(f(y),T(y))\}$, for all $x,y \in X$, $x \neq y$.

Then there exists $z \in X$ such that fz=Tz.

Further if f and T are coincidentally commutating at z then z is the unique common fixed point of f and T.

In this chapter we have generalized and extended the previous results by increasing the number of maps.

2. PRELIMINARIES:

Definition (2.1): An ultrametric space is a set X together with a function $d:X\mathrm{x}X \to R_+$, which satisfies for all x,y and z in X

(U_1) $d(x,y) \geq 0$

(U_2) $d(x,y) = 0$ if $x=y$

(U_3) $d(x,y) = d(y,x)$ (Symmentry)

(U_4) $d(x,z) \leq \max\{d(x,y),d(y,z)\}$ (strong triangle or ultrametric inequality)

Example (2.2): The discrete metric is an ultrametric.

Example (2.3): the p-adic number form a complete ultrametric space.

Definition (2.4): An ultrametric space (X,d) is said to be spherically complete if every shrinking collection of balls in X has non-empty intersection.

Definition (2.5): A self mapping T of a metric (resp. an ultrametric) space X is said to be contractive (or, strictly contractive) mapping if

$$d(Tx,Ty) < d(x,y) \text{ for all } x,y \in X \text{ with } x \neq y.$$

Example (2.6): Let $X=(-\infty,\infty)$ endowed with the usual metric and $T:X \rightarrow X$ defined by

$$Tx = x + \frac{1}{1+e^{x}}$$

for all $x \in X$. Here X is complete and T is a contractive mapping but T does not have a fixed point.

Definition (2.7): For $x \in X$, $r>0$, $B_r(x) = \{y \in X : d(x,y)<r\}$ is called the ball (open) with centre x and radius r.

3. MAIN RESULTS

Theorem (3.1): Let (X,d) be a sphereically complete ultrametric space. If $S,T:X \rightarrow X$ are mappings such that:

(i) $d(Tx,Ty)<\max\{d(Sx,Sy),d(Sx,TSx),d(Sy,TSy)\}$ $\forall$ $x,y \in X$, $x \neq y$.

(ii) $d(Sx,Sy) < d(x,y)$

(iii) $TS(x) = ST(x)$ $\forall$ $x \in X$.

Then S and T have a unique common fixed point in X.

Proof: Using condition (ii) and (iii) in (i) we have

$d(Tx,Ty) < \max\{d(Sx,Sy),d(Sx,STx),d(Sy,STy)\}$

or, $d(Tx,Ty) < \max\{d(x,y),d(x,Tx),d(y,Ty)\}$

By theorem (1.2) T has a unique fixed point i.e. z=Tz.

Now,

$$\begin{aligned} d(z,Sz) = d(Tz,STz) &= d(Tz,TSz) \\ &< \max\{d(Sz,S^2z),d(Sz,TS^2z),d(S^2z,TS^2z)\} \\ &= \max\{d(Sz,S^2z),d(Sz,S^2Tz),d(S^2z,S^2Tz)\} \end{aligned}$$

$< \max\{d(z,Sz),d(z,Sz),d(Sz,Sz)\}$

i.e. $d(z,Sz) < d(z,Sz)$, which is a contradiction and hence $z=Sz$.

Uniqueness: If possible let z and w be two distinct fixed point of S and T, then,

$$d(z,w) = d(Tz,Tw) < \max\{d(Sz,Sw),d(Sz,TSw),d(Sz,TSw)\}$$
$$= \max\{d(z,w),d(z,w),d(z,w)\}$$

i.e. $d(z,w) < d(z,w)$ which is not possible and hence $z = w$. So z is the unique fixed point of S and T.

Remarks (3.2): If we put S=I (identity map), theorem (3.1) reduces to the theorem (1.2) given by Gajic[7]

Theorem (3.3): Let (X,d) be a spherically complete metric space. If T, f and g are self maps on X satisfying:

(i) $g(x) \subseteq f(x)$

(ii) $d(g(x),g(y) < \max\{d(f(Tx),f(Ty)),d(f(Tx),g(Tx)),d(f(Ty),g(Ty))\}\ \forall\ x,y \in X,\ x\neq y$.

(iii) $d(Tx,Ty) < d(x,y)$

(iv) $T(f(x)) = f(T(x))$ and $T(g(x)) = g(Tx)$. $\forall\ x \in X$.

Then $Tz = fz = gz$. Further if f & g are commutative then there exists a unique common fixed point of T, f and g.

Proof: using condition (iii) and (iv), condition (ii) becomes

$d(g(x),g(y)) < \max\{d(T(fx),T(fy)),d(T(fx),T(gx)),d(T(fy),T(gy))\}$

or $d(g(x),g(y)) < \max\{d(fx,fy),d(fx,gx),d(fy,gy)\}$

By theorem (1.3) z is the unique common fixed point of f and g i.e. $z=f(z)=g(z)$

Now

$$d(z,Tz) = d(gz,Tgz) = d(gz,gTz)$$
$$< \max\{d(fTz,fT^2z),d(fT(z),gT(z)),d(fTz,gT^2z)\}$$
$$d(Tfz,Tgz) < \max\{d(fz,Tfz),d(fz,gz),d(fz,Tgz)\}$$
$$=\max\{d(z,Tz),d(z,z),d(z,Tz)\}$$

i.e. $d(z,Tz) < d(z,Tz)$ which is a contradiction hence $z=Tz$.

And using theorem (1.1) z is unique fixed point for T. Also from (ii) uniqueness of z follows. Hence T, f and g have unique common fixed point.

Remarks (3.4) if we put T=I, identify map, theorem (3.3) reduces to theorem (1.3) due to Rao and Kishore [5] i.e.

$$d(g(x),g(y))< \max\{d(f(x),f(y)),d(f(x),g(x)),d(f(y),g(y))\}$$

REFERENCES:

[1] A.C.M.Van Roovij: Non- Archimedean Functional Analysis, Marcel Dekker, New York, 1978.

[2] C. Petalas and T. Vidalis: A fixed point theorem in non-Archimedean vector spaces, Proc. Amer. Math. Soc. 118 (1993) 819–821.

[3] Elvin Rada: Some fixed point theorems for cyclic Contractions in Ultrametric spaces. ICT Innovations 2012 Web Proceedings ISSN 1857-7288

[4] K. P. R. Rao & G. N. V. Kishore: Common Fixed Point Theorems In Ultra Metric Spaces. Punjab University Journal of Mathematics (ISSN 1016-2526) Vol. 40 (2008) pp. 31-35.

[5]K. P. R. Rao, G. N. V. Kishore: Some Coincidence Point Theorems in Ultra and T. Ranga Rao, Metric Spaces, International Journal of Mathematical Analysis, Vol. 1, No. 18, 2007, pp. 897-902.

[6] K. P. R. Rao &G. N. V. Kishore: A Unique Common Fixed Point Theorem for two maps under ψ - φ Contractive Conditions In Ultra Metric Spaces. Mathematical Theory and Modelling, Vol. 1, No. 2, 2011

[7] Lj. Gajic: On ultra metric spaces, Novi Sad J.Math.31, 2 (2001), 69-71.

[8] Lj.Gajic: A multi valued fixed point theorem in ultra metric spaces, Math.vesnik, 54, (3-4) (2002), 89-91.

[9] Parin Chaipunya, Yeol Je Cho: Fixed Point and Common Fixed Point Wutiphol Sintunavarat, Theorems for Cyclic Quasi-Contraction & Poom Kumam in Metric and Ultrametric Spaces. Advances in Pure Mathematics, 2012, 2, 401-407. (http://www.SciRP.org/journal/apm)

[10] Qiulin Wang, Meimei Song: Common Fixed Point Theorems of Multi-Valued Maps in Ultra Metric Space. Applied Mathematics, 2013, 4, 417-420. (http://www.scirp.org/journal/am)

[11] Qiulin Wang, Meimei Song: Some Coupled Fixed Point Theorems in Ultrametric Spaces. Scientific Journal of Mathematics Research, August,2013,Vol. 4, pp 114-

[12] Rajendra Pant: Some new fixed point theorems for Contractive and non expansive mappings. Filomat 28:2 (2014), 313–317 DOI 10.2298/FIL1402313P.

[13] S.N.Mishra , Rajendra Pant: Generalization Of some fixed point theorems in ultra metric spaces. Adv. Fixed Point Theory, 4 (2014), No.1, 41-47.

[14] W. A. Kirk and N. Shahzad: Some fixed point results in ultrametric spaces, Topology Appl. 159 (2012) no. 15, 3327–3334.