Some fixed point results in ultrametric spaces

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Abstract:  
In the present paper, we obtain some new fixed point theorems for relative contractive mappings in the setting of ultrametric spaces. Our theorems complement, generalize and extend some well known results of Gajic [7], Rao and Kishore[5] and others.

Key words: fixed point theorems, ultrametric spaces

1. INTRODUCTION:

There have been a lot of generalizations of metric space such as cone metric space, G-metric space, b-metric space, probabilistic metric space, Ultrametric space etc. One of the recent generalization is ultrametric space defined by Rooji [1]. After the introduction of ultrametric space, Petals and Vidalis [2] proved a fixed point theorem for contractive mappings on spherically complete ultrametric space X.

Petals and Vidalis [2] established the following fixed point theorem:
Theorem (1.1): Let \((X,d)\) be a spherically complete ultrametric space and \(T:X\rightarrow X\) a contractive mapping. Then \(T\) has a unique fixed point.

In 2001 Gajic [7] obtained the following generalization of the above theorem:

**Theorem (1.2):** Let \((X,d)\) be a spherically complete ultrametric space and \(T:X\rightarrow X\) a mapping such that for all \(x,y \in X, x\neq y,\)
\[d(Tx,Ty) < \max \{ d(x,y), d(x,Tx), d(y,Ty) \}\]
Then \(T\) has a unique fixed point.

Later on Rao and Kishore [5] extended the above result for a pair of maps of Jungck type as follows:

**Theorem (1.3):** Let \((X,d)\) be a spherically complete ultrametric space. If \(f\) and \(T\) are self maps on \(X\) satisfying
\[d(Tx,Ty) < \max \{ d(f(f(x),f(y)), d(f(x),T(x)),d(f(y),T(y)) \}, \quad \text{for all } x,y \in X, x \neq y.\]
Then there exists \(z \in X\) such that \(fz=Tz.\)
Further if \(f\) and \(T\) are coincidentally commutating at \(z\) then \(z\) is the unique common fixed point of \(f\) and \(T.\)

In this chapter we have generalized and extended the previous results by increasing the number of maps.

2. PRELIMINARIES:

**Definition (2.1):** An ultrametric space is a set \(X\) together with a function \(d:XxX \rightarrow \mathbb{R}_{+}\), which satisfies for all \(x,y\) and \(z\) in \(X\)
\[(U_1) \quad d(x,y) \geq 0\]
\[(U_2) \quad d(x,y) = 0 \text{ if } x=y\]
\[(U_3) \quad d(x,y) = d(y,x) \text{ (Symmetry)}\]
\[(U_4) \quad d(x,z) \leq \max \{d(x,y),d(y,z)\} \text{ (strong triangle or ultrametric inequality)}\]

**Example (2.2):** The discrete metric is an ultrametric.
**Example (2.3):** the p-adic number form a complete ultrametric space.

**Definition (2.4):** An ultrametric space \((X, d)\) is said to be spherically complete if every shrinking collection of balls in \(X\) has non-empty intersection.

**Definition (2.5):** A self mapping \(T\) of a metric (resp. an ultrametric) space \(X\) is said to be contractive (or, strictly contractive) mapping if
\[
d(Tx, Ty) < d(x, y) \quad \text{for all } x, y \in X \text{ with } x \neq y.
\]

**Example (2.6):** Let \(X=(-\infty, \infty)\) endowed with the usual metric and \(T: X \to X\) defined by
\[
Tx = x + \frac{1}{1 + e^x}
\]
for all \(x \in X\). Here \(X\) is complete and \(T\) is a contractive mapping but \(T\) does not have a fixed point.

**Definition (2.7):** For \(x \in X\), \(r > 0\), \(B_r(x) = \{y \in X : d(x, y) < r\}\) is called the ball (open) with centre \(x\) and radius \(r\).

### 3. MAIN RESULTS

**Theorem (3.1):** Let \((X, d)\) be a spherically complete ultrametric space. If \(S, T: X \to X\) are mappings such that:

(i) \(d(Tx, Ty) < \max\{d(Sx, Sy), d(Sx, TSx), d(Sy, TSy)\} \quad \forall x, y \in X, x \neq y\).

(ii) \(d(Sx, Sy) < d(x, y)\)

(iii) \(TS(x) = ST(x) \quad \forall x \in X\).

Then \(S\) and \(T\) have a unique common fixed point in \(X\).

**Proof:** Using condition (ii) and (iii) in (i) we have
\[
d(Tx, Ty) < \max\{d(Sx, Sy), d(Sx, STx), d(Sy, STy)\}
\]
or,
\[
d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty)\}
\]
By theorem (1.2) \(T\) has a unique fixed point i.e. \(z = Tz\).

Now,
\[
d(z, Sz) = d(Tz, STz) = d(Tz, TSz)
\]
\[
< \max\{d(Sz, S^2z), d(Sz, TS^2z), d(S^2z, TS^2z)\}
\]
\[
= \max\{d(Sz, S^2z), d(Sz, S^2Tz), d(S^2z, S^2Tz)\}
\]
< \max\{d(z,Sz),d(z,Sz),d(Sz,Sz)\}
i.e. \ d(z,Sz) < d(z,Sz), which is a contradiction and hence z=Sz.

Uniqueness: If possible let z and w be two distinct fixed point of S and T, then,
\[d(z,w) = d(Tz,Tw) < \max\{d(Sz,Sw),d(Sz,TSw),d(Sz,TSw)\}\]
= \max\{d(z,w),d(z,w),d(z,w)\}
i.e. d(z,w) < d(z,w) which is not possible and hence z = w. So z is the unique fixed point of S and T.

Remarks (3.2): If we put S=I (identity map), theorem (3.1) reduces to the theorem (1.2) given by Gajic[7]

Theorem (3.3): Let (X,d) be a spherically complete metric space. If T, f and g are self maps on X satisfying:
(i) \ g(x) \subseteq f(x)
(ii) \ d(g(x),g(y))< \max\{d(f(Tx),f(Ty)),d(f(Tx),g(Tx)),d(f(Ty),g(Ty))\} \forall \ x,y \in X, x \neq y.
(iii) \ d(Tx,Ty) < d(x,y)
(iv) \ T(f(x)) = f(T(x)) \text{ and } T(g(x)) = g(Tx). \forall \ x \in X.

Then Tz = fz = gz. Further if f & g are commutative then there exists a unique common fixed point of T, f and g.

Proof: using condition (iii) and (iv), condition (ii) becomes
\[d(g(x),g(y)) < \max\{d(T(fx),T(fy)),d(T(fx),T(gx)),d(T(fy),T(gy))\}\]
or \[d(g(x),g(y)) < \max\{d(fx,fy),d(fx,gx),d(fy,gy)\}\]
By theorem (1.3) z is the unique common fixed point of f and g
i.e. z=f(z)=g(z)

Now
\[d(z,Tz) = d(gz,Tgz) = d(gz,gTz)\]
< \max\{d(fTz,fT^2z),d(fT(z),gT(z)),d(fTz,gT^2z)\}\]
\[d(Tfz,Tgz) = \max\{d(fz,Tfz),d(fz,gz),d(fz,Tgz)\}\]
i.e. d(z,Tz) < d(z,Tz) which is a contradiction hence z=Tz.
And using theorem (1.1) z is unique fixed point for T. Also from (ii) uniqueness of z follows. Hence T, f and g have unique common fixed point.
Remarks (3.4) if we put \(T=I\), identify map, theorem (3.3) reduces to theorem (1.3) due to Rao and Kishore [5] i.e.
\[d(g(x),g(y)) \leq \max\{d(f(x),f(y)),d(f(x),g(x)),d(f(y),g(y))\}\]

REFERENCES: