Coupled Fixed Point Theorem on Partially Ordered G-metric Space

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Abstract:
Ayedi et al. established coupled coincidence and coupled common fixed point result. Recently Erdal Karapinar, Billur Kayamakcalan and Kenan Tas [19] improved and extend the coupled fixed point of Ayedi et al.[2] Now we prove some recent coupled fixed point theorem in partially ordered G-metric spaces.

Key words: Coupled Fixed Point Theorem, Partially Ordered G-metric Space

INTRODUCTION AND PRELIMINARY:

One of the simplest and the most useful result in the fixed point theory is a Banach Contraction Principal [6]. These principal has been generalized in different direction in different spaces by mathematicians.

In [2] Ayedi et al. established coupled coincidence and coupled common fixed point results for a mixed g-monotone mapping satisfying Non-linear contraction in partially ordered G-metric spaces. These result generalize those of Choudhary
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and Maity [9]. Consequently Erdal Karapinar, Billur Kaymakcalan and Kenan Tas improved the result of Ayedi et al.

**Definition 1.1** Let X be a non-empty set, and G : X × X × X → R+ be a function satisfying the following properties:

(G1) G(x, y, z) = 0, if x = y = z,

(G2) 0 < G(x, x, y) for all x, y ∈ X with x ≠ y,

(G3) G(x, y, z) ≤ G(x, x, x) for all x, y, z ∈ X with y ≠ z,

(G4) G(x, y, z) = G(x, z, y) = G(y, z, x) = · · · (symmetry in all three variables),

(G5) G(x, y, z) ≤ G(x, a, a) + G(a, y, z) for all x, y, z, a ∈ X (rectangle inequality).

Then the function G is called a generalized metric or, more specially, a G-metric on X, and the pair (X,G) is called a G-metric space.

Every G-metric on X defines a metric dG on X by

\[ dG(x, y) = G(x, y, y) + G(y, x, x), \]

for all x, y ∈ X. (1.1)

**Example 1.2** Let (X, d) be a metric space. The function G: X×X×X→[0,+∞), defined by

\[ G(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\} \]

or

\[ G(x, y, z) = d(x, y) + d(y, z) + d(z, x), \]

for all x, y, z ∈ X, is a G-metric on X.

**Definition 1.3** Let (X,G) be a G-metric space, and let \( \{x_n\} \) be a sequence of points of

We say that \( (x_n) \) is G-convergent to x ∈ X if \( \lim_{n,m \to +\infty} G(x, x_n, x_m) = 0 \), that is, for any ε > 0, there exists N ∈ N such that \( G(x, x_n, x_m) < \varepsilon \), for all n, m ≥ N. We call x the limit of the sequence and write \( x_n \to x \) or \( \lim_{n \to +\infty} x_n = x \).

**Proposition 1.4** Let (X,G) be a G-metric space. The following are equivalent:
(1) \( \{x_n\} \) is \( G \)-convergent, to \( x \)
(2) \( G(x_n, x_n, x) \to 0 \) as \( n \to +\infty \),
(3) \( G(x_n, x, x) \to 0 \) as \( n \to +\infty \),
(4) \( G(x_n, x_n, x) \to 0 \) as \( n, m \to +\infty \).

**Definition 1.5** Let \((X, G)\) be a \( G \)-metric space. A sequence \( \{x_n\} \) is called a \( G \)-Cauchy sequence if, for any \( \varepsilon > 0 \), there is \( N \in \mathbb{N} \) such that \( G(x_n, x_m, x_l) < \varepsilon \) for all \( m, n, l \geq N \), that is, \( G(x_n, x_m, x_l) \to 0 \) as \( n, m, l \to +\infty \).

**Proposition 1.6** Let \((X, G)\) be a \( G \)-metric space. Then the following are equivalent:
(1) The sequence \( \{x_n\} \) is \( G \)-Cauchy,
(2) For any \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that \( G(x_n, x_m, x_l) < \varepsilon \), for all \( m, n \geq N \).

**Proposition 1.7** Let \((X, G)\) be a \( G \)-metric space. A mapping \( f : X \to X \) is \( G \)-continuous at \( x_0 \) if and only if it is \( G \)-sequentially continuous at \( x_0 \), that is, whenever \( \{x_n\} \) is \( G \)-convergent to \( x_0 \), the sequence \( \{f(x_n)\} \) is \( G \)-convergent to \( f(x_0) \).

**Definition 1.8** A \( G \)-metric space \((X, G)\) is called \( G \)-complete if every \( G \)-Cauchy sequence is \( G \)-convergent in \((X, G)\).

**Definition 1.9** Let \((X, G)\) be a \( G \)-metric space. A mapping \( F : X \times X \to X \) is said to be continuous if for any two \( G \)-convergent sequences \( \{x_n\} \) and \( \{y_n\} \) converging to \( x, y \) respectively, \( \{F(x_n, y_n)\} \) is \( G \)-convergent to \( F(x, y) \).

Let \((X, \leq)\) be a partially ordered set and \((X, G)\) be a \( G \)-metric space, \( g : X \to X \) be a mapping.

A partially ordered \( G \)-metric space, \((X, G, \leq)\), is called \( g \)-ordered complete if for each convergent sequence \( \{x_n\}_{n=0}^\infty \subseteq X \), the following conditions hold:
(1) if \( \{x_n\} \) is a non-increasing sequence in \( X \) such that \( x_n \to x \) implies \( gx \leq gx_n \) for all \( n \in \mathbb{N} \),
(2) if \( \{y_n\} \) is a non-decreasing sequence in \( X \) such that \( y_n \to y \) implies \( gy \geq g y_n \), \( \forall n \in \mathbb{N} \).

Moreover, a partially ordered \( G \)-metric space, \((X,G, \leq)\), is called ordered complete when \( g \) is equal to identity mapping in the above conditions (1) and (2).

**Definition 1.10** An element \((x, y) \in X \times X\) is said to be a coupled fixed point of the mapping \( F : X \times X \to X \) if \( F(x, y) = x \) and \( F(y, x) = y \).

**Definition 1.11** An element \((x, y) \in X \times X\) is called a coupled coincidence point of a mapping \( F : X \times X \to X \) and \( g : X \to X \) if \( F(x, y) = g(x) \), \( F(y, x) = g(y) \).

Moreover, \((x, y) \in X \times X\) is called a common coupled coincidence point of \( F \) and \( g \) if \( F(x, y) = g(x) = x \), \( F(y, x) = g(y) = y \).

**Definition 1.12** Let \( F : X \times X \to X \) and \( g : X \to X \) be mappings. The mappings \( F \) and \( g \) are said to commute if \( g(F(x, y)) = F(g(x), g(y)) \), for all \( x, y \in X \).

**Definition 1.13** Let \((X, \leq)\) be a partially ordered set and \( F : X \times X \to X \) be a mapping.

Then \( F \) is said to have mixed monotone property if \( F(x, y) \) is monotone non-decreasing in \( x \) and is monotone non-increasing in \( y \), that is, for any \( x, y \in X \),

\[ x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y), \text{ for } x_1, x_2 \in X, \]

And \( y_1 \leq y_2 \Rightarrow F(x, y_2) \leq F(x, y_1) \), for \( y_1, y_2 \in X \).

**Definition 1.14** Let \((X, \leq)\) be a partially ordered set and \( F : X \times X \to X \) and \( g : X \to X \) be two mappings. Then \( F \) is said to have mixed \( g \)-monotone property if \( F(x, y) \) is monotone \( g \)-non-decreasing in \( x \) and is monotone \( g \)-non-increasing in \( y \), that is, for any \( x, y \in X \),

\[ g(x_1) \leq g(x_2) \Rightarrow F(x_1, y) \leq F(x_2, y), \text{ for } x_1, x_2 \in X, \] and \( (1.2) \)

\[ g(y_1) \leq g(y_2) \Rightarrow F(x, y_2) \leq F(x, y_1), \text{ for } y_1, y_2 \in X. \] (1.3)
Let $\emptyset$ denote the set of functions $\emptyset^{-1} : [0, \infty) \rightarrow [0, \infty)$ satisfying
\begin{align*}
& (a) \quad \emptyset^{-1}(\{0\}) = \{0\}, \\
& (b) \quad \emptyset(t) < t \text{ for all } t > 0, \\
& (c) \quad \lim_{r \to t^+} \emptyset(r) < t \text{ for all } t > 0.
\end{align*}

**Main Result:**

**Theorem 2.1** Let $(X, \preceq)$ be a partially ordered set and $G$ be a G-metric on $X$ such that $(X, G)$ is a complete G-metric. Suppose that there exist $\Phi \in \Phi$, $f : X \times X \rightarrow X$ and $g : X \rightarrow X$ such that
\[ G(f(x, y), f(u, v), f(w, z)) + G(f(y, x), f(v, u), f(z, w)) \leq G(g(x, y), g(u, v), g(w, z)) - \Phi G(g(x, y), g(u, v), g(w, z)) \tag{2.1} \]

For all $x, y, u, v, w, z \in X$ with $g(x) \preceq g(y) \preceq g(z)$, suppose also that $f$ is continuous and has the mixed g-monotone property, $f(X \times X) \subseteq g(X)$ and $g$ is continuous and commutes with $f$. If there exist $x_0, y_0 \in X$ such that $g(x_0) \preceq f(x_0, y_0)$ and $f(y_0, x_0) \preceq g(y_0)$, then $f$ and $g$ have a coupled coincident point, that is there exist $(x, y) \in X \times X$ such that $gx = f(x, y)$ and $gy = f(y, x)$

**Proof:** Given $x_0, y_0 \in X$ satisfying $gx_0 \preceq f(x_0, y_0)$ and $f(y_0, x_0) \preceq g(y_0)$, we shall construct iterative sequence $(x_n)$ and $(y_n)$ in the following way: $f(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = f(x_0, y_0)$ and $gy_1 = f(y_0, x_0)$. Similarly we can choose $x_2, y_2 \in X$ such that $gx_2 = f(x_1, y_1)$ and $gy_2 = f(y_1, x_1)$. Since $f$ has the mixed $g$-monotone property, we conclude that $gx_0 \preceq gx_1 \preceq gx_2$ and $gy_0 \preceq gy_1 \preceq gy_2$. We get from above
\[ gx_n = f(x_{n-1}, y_{n-1}) \leq gx_{n+1} = f(x_n, y_n) \quad \text{and} \quad gy_{n+1} = f(y_n, x_n) \leq gy_n = f(y_{n-1}, x_{n-1}) \]

If for some $n_0$ we have $(gx_{n_0+1}, gy_{n_0+1}) = (gx_{n_0}, gy_{n_0})$, then
\[ f(x_{n_0}, y_{n_0}) = gx_{n_0} \text{ and } f(y_{n_0}, x_{n_0}) = gy_{n_0} \]

that is $f$ and $g$ have a coincidence point. So we assume that $(gx_{n+1}, gy_{n+1}) \neq (gx_n, gy_n)$ for all $n \in \mathbb{N}$, Thus we have either
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We define \( s_n = G(gx_{n+1}, gx_n, gx_n) + G(gy_{n+1}, gy_n, gy_n) \) \((2.2)\)
for all \( n \in \mathbb{N} \). Due to the property (G2), we have \( s_n > 0 \) for all \( n \in \mathbb{N} \).

By using inequality (2.1),
\[
G(gx_{n+1}, gx_n, gx_n) + G(gy_{n+1}, gy_n, gy_n) = G(f(x_n, y_n), f(x_{n-1}, y_{n-1}), f(gx_{n-1}, gy_{n-1})) \\
+ G(f(y_n, x_n), f(y_{n-1}, x_{n-1}), f(gy_{n-1}, gx_{n-1})) \\
\leq [G(gx_n, gx_{n-1}, gx_{n-1}) + G(gy_n, gy_{n-1}, gy_{n-1})] \\
- \theta[G(gx_n, gx_{n-1}, gx_{n-1}) + G(gy_n, gy_{n-1}, gy_{n-1})] \quad (2.3)
\]
so that
\[
s_n \leq s_{n-1} - \theta(s_{n-1}) \quad (2.4)
\]

Since \( \theta(t) < t \) for all \( t > 0 \), it follows that \( s_n \) is monotone decreasing. Therefore, there is some \( s \geq 0 \) such that \( \lim_{n \to \infty} s_n = s \).

Now, we assert that \( s = 0 \). Suppose, on contrary, that \( s > 0 \).

Letting \( n \to +\infty \),
\[
s = \lim_{n \to +\infty} s_n \leq \lim_{n \to +\infty} s - \theta(s) < s
\]

This is a contradiction. Thus \( s = 0 \). Hence
\[
\lim_{n \to +\infty} s_n = \lim_{n \to +\infty} G(gx_{n+1}, gx_n, gx_n) + G(gy_{n+1}, gy_n, gy_n) = 0 \quad (2.5)
\]

Next we prove that \((gx_n),(gy_n)\) are Cauchy sequence in G-metric space \((X, G)\). Suppose on contrary, that at least one of \((gx_n),(gy_n)\) is not a Cauchy sequence in \((X, G)\). Then there exists \( \epsilon > 0 \) and sequence of natural number \((m_k),(n_k)\) such that for every natural number \( k \), \( (m_k) > (n_k) \geq k \) and
\[
r_k = G(gx_{m_k}, gx_{m_k}, gx_{n_k}) + G(gy_{m_k}, gy_{n_k}, gy_{n_k}) \geq \epsilon \quad (2.6)
\]

Now corresponding to \((n_k)\), we choose \((m_k)\) to be smallest for which (2.6) holds. Hence
\[
G(gx_{m_k-1}, gx_{n_k}, gx_{n_k}) + G(gy_{m_k-1}, gy_{n_k}, gy_{n_k}) < \epsilon
\]

Using rectangular inequality and \( G_5 \), we get
\[
\epsilon \leq r_k \\
\leq G(gx_{m_k}, gx_{m_k-1}, gx_{m_k-1}) + G(gx_{m_k-1}, gx_{n_k}, gx_{n_k}) \\
+ G(gy_{m_k}, gy_{m_k-1}, gy_{m_k-1}) + G(gy_{m_k-1}, gy_{n_k}, gy_{n_k}) \\
= G(gx_{m_k-1}, gx_{n_k}, gx_{n_k}) + G(gy_{m_k-1}, gy_{n_k}, gy_{n_k}) + s_{m_k-1} \\
< \epsilon + s_{m_k-1} \quad (2.7)
\]
Letting $k \to +\infty$ in the above inequality and using (2.6), we get
\[ \lim_{k \to \infty} r_k = \epsilon \quad (2.8) \]
Again, by rectangle inequality, we have
\[
\begin{align*}
r_k &= G\left( g x_{m_k} g x_{m_k}, g x_{n_k} \right) + G\left( g y_{m_k} g y_{n_k} g y_{n_k} \right) \\
& \leq G\left( g x_{m_k} g x_{m_k+1}, g x_{m_k+1} \right) + G\left( g x_{m_k+1} g x_{m_k+1}, g x_{n_k} \right) + \\
& \quad + G\left( g y_{m_k} g y_{m_k+1}, g y_{n_k+1} \right) + G\left( g y_{m_k+1} g y_{n_k+1}, g y_{n_k+1} \right) \\
& = s_{n_k} + 2 G\left( g x_{m_k+1}, g x_{n_k+1} \right) + G\left( g y_{m_k+1}, g y_{n_k+1} \right) + G\left( g y_{m_k}, g y_{n_k+1}, g y_{n_k+1} \right) \\
& \quad + G\left( g x_{m_k+1}, g x_{n_k+1}, g x_{n_k} \right) + G\left( g y_{m_k+1}, g y_{n_k+1}, g y_{n_k} \right)
\end{align*}
\]
Using the fact that $G(x, y, z) \leq 2G(y, x, x)$ for any $x, y \in X$, we obtain
\[
\begin{align*}
r_k & \leq s_{n_k} + 2 G\left( g x_{m_k} g x_{m_k}, g x_{m_k+1} \right) + 2 G\left( g y_{m_k} g y_{m_k}, g y_{m_k+1} \right) \\
& \quad + G\left( g x_{m_k+1} g x_{n_k+1}, g x_{n_k}, g y_{m_k+1} \right) + G\left( g y_{m_k+1} g y_{n_k+1}, g y_{n_k+1}, g y_{n_k+1} \right) \\
& \quad + G\left( g x_{m_k+1}, g x_{n_k+1}, g x_{n_k} \right) + G\left( g y_{m_k+1}, g y_{n_k+1}, g y_{n_k} \right)
\end{align*}
\]
Next, Using inequality (2.1), we have
\[
\begin{align*}
& G\left( g x_{m_k+1} g x_{n_k+1}, g x_{n_k+1} \right) + G\left( g y_{m_k+1} g y_{n_k+1}, g y_{n_k+1} \right) \\
& = G\left( f(x_{m_k+1} g x_{n_k}), f(x_{n_k} g y_{n_k}), f(x_{n_k} g y_{n_k}) \right) + G\left( f(y_{n_k} g x_{n_k}), f(y_{n_k} g x_{n_k}), f(y_{n_k} g y_{n_k}) \right) \\
& \leq G\left( g x_{m_k} g x_{n_k}, g x_{n_k} \right) + G\left( g y_{m_k} g y_{n_k} g y_{n_k} \right) \\
& \quad - \phi\left( G\left( g x_{m_k} g x_{n_k}, g x_{n_k} \right) + G\left( g y_{m_k} g y_{n_k} g y_{n_k} \right) \right)
\end{align*}
\]
\[
\leq r_k - \phi(r_k) \\
(2.9)
\]
By using (2.5), (2.8) and letting $k \to \infty$, we get,
\[ \epsilon \leq \lim_{k \to \infty} r_k - \phi(r_k) < \epsilon \]
This is contradiction. So $(g x_n), (g y_n)$ are Cauchy sequence in G metric space $(X, G)$. Since $(X, G)$ is complete then there exist $x, y \in X$ such that $(g x_n)$ and $(g y_n)$ are $G$-convergent to $x$ and $y$.

From proposition 1.4, we have
\[
\begin{align*}
\lim_{n \to \infty} G(g x_n, x, x) &= 0 \quad \text{and} \quad \lim_{n \to \infty} G(g y_n, y, y) = 0
\end{align*}
\]
Using continuity of $g$, we get from proposition 1.7,
\[
\begin{align*}
\lim_{n \to \infty} G(g(g x_n), g x, g x) &= 0 \quad \text{and} \quad \lim_{n \to \infty} G(g(g y_n), g y, g y) = 0 \\
(2.10)
\end{align*}
\]
Since $g x_{n+1} = f(x_n, y_n)$ and $g y_{n+1} = f(y_n, x_n)$, employing the commutativity of $f$ and $g$,
\[
\begin{align*}
g(g x_{n+1}) &= g(f(x_n, y_n)) = f((g x_n, g y_n)) \\
g(g y_{n+1}) &= g(f(y_n, x_n)) = g(f(g y_n, g x_n)). \\
(2.11)
\]
Now we shall show that $f(x, y) = gx$ and $f(y, x) = gy$.
The mapping $f$ is continuous, and since the sequence $(gx_n)$ and $(gy_n)$ are respectively $G$-convergent to $x$ and $y$, Using definition 1.9, the sequence $(f(gx_n, gy_n))$ is $G$-convergent to $f(x, y)$. Therefore from (2.11), $(g(gx_{n+1}))$ is $G$-convergent to $f(x, y)$. By uniqueness of the limit and using (2.10), we have $f(x, y) = gx$. Similarly, we can show that $f(y, x) = gy$. Hence $(x, y)$ is a coupled coincidence point of $f$ and $g$. This completes the proof.

**Theorem-2.2:** Let $(X, \leq)$ be a partially ordered set and $G$ be a $G$-metric on $X$ such that $(X, G, \leq)$ is a complete $G$-metric. Suppose that there exist $\Phi \subseteq \Phi$, $f : X \times X \rightarrow X$ and $g : X \rightarrow X$ such that

$$[G(f(x, y), f(u, v), f(w, z))] + [G(f(y, x), f(v, u), f(z, w))] \leq [G(gx, gu, gw) + G(gy, gv, gz)] - \Phi [G(gx, gu, gw) + G(gy, gv, gz)] \quad (2.1)$$

For all $x, y, u, v, w, z \in X$ with $gw \leq gu \leq gx$ and $gy \leq gv \leq gz$. Suppose also $(g(x), G)$ is complete, $f$ has the mixed $g$-monotone property, $f(X \times X) \subseteq g(x)$. If there exist $x_0, y_0 \in X$ such that $gx_0 \leq f(x_0, y_0)$ and $f(y_0, x_0) \leq gy_0$, then $f$ and $g$ have a coupled coincidence point.

**Proof:** proceeding exactly as in Theorem 2.1. We have $(gx_n)$ and $(gy_n)$ are Cauchy sequence in the complete $G$-metric spaces $(g(x), G)$. Then there exist $x, y \in X$ such that $gx_n \rightarrow gx$ and $gy_n \rightarrow gy$.

Since $(gx_n)$ is non-decreasing and $(gy_n)$ is non-increasing, then we have $gx_n \leq gx$ and $gy_n \leq gy$ for all $n \geq 0$. If $gx_n = gx$ and $gy_n = gy$ for some $n \geq 0$, then $gx = gx_n \leq gx_{n+1} \leq gx = gx_n$ and $gy = gy_{n+1} \leq gy_n \leq gy$, which implies that $gx_n = gx_{n+1} = f(x_n, y_n)$ and $gy_n = gy_{n+1} = f(y_n, x_n)$, that is a coupled coincidence point of $f$ and $g$. Then we assume that $g(x_n, y_n) \neq (gx, gy)$ for all $n \geq 0$. 

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Then by rectangle inequality, we have
\[
G(f(x, y), gx, gx) + G(f(y, x), gy, gy) \leq G(f(x, y), gx_{n+1}, gx_{n+1}) + G(gx_{n+1}, gx, gx) \\
+ G(f(y, x), gy_{n+1}, gy_{n+1}) + G(gy_{n+1}, gy, gy) \\
= G(f(x, y), f(x_n, y_n), f(x_n, y_n)) + G(gx_{n+1}, gx, gx) \\
+ G(f(y, x), f(y_n, x_n), f(y_n, x_n)) + G(gy_{n+1}, gy, gy) \\
\leq \{G(gx, gx_{n}, gx_{n}) + G(gy, gy_{n}, gy_{n})\} + \\
\{G(gx_{n+1}, gx, gx) + G(gy_{n+1}, gy, gy)\} \\
- \Phi\{G(gx, gx_{n}, gx_{n}) + G(gy, gy_{n}, gy_{n})\} + \\
\{G(gx_{n+1}, gx, gx) + G(gy_{n+1}, gy, gy)\}
\]

As \( n \to \infty \) in above inequality, we have
\[
G(f(x, y), gx, gx) + G(f(y, x), gy, gy) = 0,
\]
Which implies that \( gx = f(x, y) \) and \( gy = f(y, x) \). Hence \((x, y)\) is a coupled coincident point of \( f \) and \( g \).

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