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Inverse of Al-Tememe Transform

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Abstract:

Our aim in this paper is to compute the inverse of Al-Tememe transform by using complex variable and by using residue theorem we find the inverse .

Introduction: *We can use complex variables to compute the inverse of Al-Tememe Transform for a continuous function* $f(x)$ *and using complex inversion formula of Al-Tememe Transform by consider our variable* p *to be a complex variable* z *where we find that complex inversion formula equal to the residue by residue theorem .*

Key words: AL-Tememe Transform, variable coefficients, partial fraction, inverse of Al-Tememe transform, Analytic functions, Cauchy's Integral Formula, piecewise continuous function, the complex integral

BASIC DEFINITIONS AND CONCEPTS:

Towards make the work is self-contained as much as possible, we will start by introducing some of the important definitions and concepts that used then within of the search.

Definition 1: **[1]**

Let f is defined function at period (a, b) then the integral transformation for f whose it's symbol $F(p)$ is defined as:

$$F(p) = \int_a^b k(p,x)\, f(x)\, dx$$

Where k is a fixed function of two variables, called the kernel of the transformation, and a, b are real numbers or ∓∞, such that the above integral converges.

Definition 2: **[3]**

The Al-Tememe transformation for the function $f(x)$; $x > 1$ is defined by the following integral:

$$\mathcal{T}\,[f(x)] = \int_1^\infty x^{-p} f(x)\, dx = F(p)$$

such that this integral is convergent , p is positive constant

Property 1: **[3]**

This transformation is characterized by the linear property, that is

$$\mathcal{T}\,[Af(x) + Bg(x)] = A\mathcal{T}[f(x)] + B\mathcal{T}[g(x)]\,,$$

Where, are constants, the functions $f(x)\,, g(x)$ are defined when $x > 1$.

The Al-Tememe transform for some fundamental functions are given in table(1) **[3]** :

ID	Function, $f(x)$	$F(p) = \int_1^\infty x^{-p} f(x)dx = \mathcal{T}\,[f(x)]$	Regional of convergence
1	k ; k =constant	$\frac{k}{p-1}$	$\boldsymbol{p > 1}$
2	$x^n, n \in R$	$\frac{1}{p-(n+1)}$	$\boldsymbol{p > n + 1}$

3	lnx	$\frac{1}{(p-1)^2}$	$\boldsymbol{p>1}$
4	$x^n lnx, n \in R$	$\frac{1}{[p-(n+1)]^2}$	$\boldsymbol{p>n+1}$
5	$sin(alnx)$	$\frac{a}{(p-1)^2+a^2}$	$\boldsymbol{p>1}$ a =constant
6	$cos(alnx)$	$\frac{p-1}{(p-1)^2+a^2}$	$\boldsymbol{p>1}$ a =constant
7	$sinh(alnx)$	$\frac{a}{(p-1)^2-a^2}$	$\boldsymbol{\lvert p-1\rvert > a}$ a =constant
8	$cosh(alnx)$	$\frac{p-1}{(p-1)^2-a^2}$	$\boldsymbol{\lvert p-1\rvert > a}$ a =constant

Table (1)

From the Al-Tememe definition and the above table, we get:

Theorem 1: **[3]**

If $\mathcal{T}\, f(x) = F(p)$ and a is constant, then $\mathcal{T}\, f\,(x^{-a}) = F(p+a)$

Definition 3: **[3]**

Let $f(x)$ be a function where $(x > 1)$ and $\mathcal{T}\, f(x) = F(p)$, $f(x)$ is said to be an inverse for the Al-Tememe transformation and written as $\mathcal{T}^{-1} F(p) = f(x)$, where $\mathcal{T}^{-1}$ returns the transformation to the original function.

Property 2: **[3]**

If $\mathcal{T}^{-1} F_1(p) = f_1(x)$, $\mathcal{T}^{-1} F_2(p) = f_2(x)$,…, $\mathcal{T}^{-1} F_n(p) = f_n(x)$ and $a_1, a_2, \ldots a_n$ are constants then,

$$\mathcal{T}^{-1}[a_1F_1(p) + a_2F_2(p) + \cdots + a_nF_n(p)] = a_1f_1(x) + a_2f_2(x) + \cdots + a_n f_n(x)$$

Theorem 2: **[3]**

If the function $f(x)$ is defined for $x > 1$ and its derivatives $f^{(1)}(x), f^{(2)}(x), \ldots, f^{(n)}(x)$ are exist then:

$$\mathcal{T}\left[x^n f^{(n)}(x)\right] = -f^{(n-1)}(1) - (p-n)f^{(n-2)}(1) - \cdots$$
$$-(p-n)\,(p-(n-1))\,\ldots(p-2)\,f(1) + (p-n)!\,F(p)$$

Definition 4: **[4]**

A function $f(x)$ is piecewise continuous on an interval $[a,b]$ if the interval can be partitioned by a finite number of points $a = x_0 < x_1 < \cdots < x_n = b$ such that:

1. $f(x)$ is continuous on each subinterval (x_i, x_{i+1}) , for $i = 0, 1, 2, \dots, n-1$
2. The function f has jump discontinuity at x_i , thus $\left|\lim_{x\to x_i^+} f(x)\right| < \infty$, $i = 0, 1, 2, \dots, n-1$; $\left|\lim_{x\to x_i^-} f(x)\right| < \infty$, $i = 0, 1, 2, \dots, n$

Analytic Functions: [6]

A function $f(z)$ is analytic if it is differentiable at all points of some neighborhood $|z - z_0| < r$ then $f(z)$is said to be analytic(holomorphic) at z_0 if $f(z)$ is analytic at each point of domain D then $f(z)$ is analytic in D .

Theorem (3):**[5]**

Let C denote a contour of length L, and suppose that a function $f(z)$ is piecewise continuous on C. If M is a nonnegative constant such that $|f(z)| = M$ for all points z on C at which $f(z)$ is defined , then

$$\left|\int_C f(z)\, dz\right| \le ML$$

Theorem (4):**[5]** (*JORDAN'S LEMMA*)

Suppose that ,

(a) A function $f(z)$ is analytic at all points in the upper half plane $y \ge 0$ that are exterior to a circle $|z| = R_0$,

(b) C_R denotes a semicircle $z = Re^{i\theta}$ $(0 \le \theta \le \pi)$, where $R > R_0$,

(c) For all points z on C_R, there is a positive constant M_R such that,

$$|f(z)| \le M_R \ \ and \ \ \lim_{R\to\infty} M_R = 0$$

Then , for every positive constant a ,

$$\lim_{R\to\infty} \int_{C_R} f\,(z)\, e^{iaz}\, dz = 0$$

Theorem (5): **[5]** (Cauchy's Theorem or Cauchy-Goursat Theorem)
If $f(z)$ is analytic within and on a simple (non-intersecting) closed curve C s.t. $f'(z)$ is also continuous there, then

$$\oint_C f(z)\, dz = 0$$

For multiply connected domains (MCD), we instead have

$$\oint_{C_1} f\,(z)\, dz - \oint_{C_2} f\,(z)\, dz = 0$$

Cauchy's Integral Formula:[6]
If $f(z)$ is analytic within and on a simple closed curve C, then for z_o any interior point in C,

$$\left.\frac{d^n f}{dz^n}\right|_{z=z_o} = \frac{n!}{2\pi i} \oint_{C_2} \frac{f(\omega)}{(\omega - z_o)^{n+1}}\, dz$$

Residue Theorem: [5]
Assume $f(z)$ is analytic within and on a simple closed curve C except for at isolated singularities $z_1, z_2, \dots, z_n$ within C. Draw non-intersecting circles $C_1, C_2, \dots, C_n$ all lying within C. We now have a multiply connected domain (MCD) such that $f(z)$ is analytic inside it and on its boundary. Then by the Cauchy-Goursat theorem,

$$\oint_C f\,(z)\, dz - \oint_{C_1} f\,(z)\, dz - \oint_{C_2} f\,(z)\, dz - \cdots - \oint_{C_n} f\,(z)\, dz = 0$$

$$Or \quad \oint_C f(z)\,dz - \sum_{j=1}^{n} \oint_{C_j} f(z)\,dz = 0$$

Since $z_1, z_2, \ldots, z_n$ are isolated singularities we have Laurent series and can compute the residues. Define

$$k_j = \frac{1}{2\pi i} \oint_{C_j} f(z)\,dz$$

k_j are the residues of $f(z)$ at z_j. Thus

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{n} k_j$$

Inverse of Al-Tememe Transform:

Suppose $F(p)$ is Al-Tememe transform of the piecewise continuous function $f(x)$ of exponential order, that is analytic on and the to right of the line $R(z) = a$ (see figuer 1)

Then by Cauchy 's integral formula

$$F(p) = \frac{1}{2\pi i} \oint_C \frac{F(z)}{z-p}\,dz$$

$$= \frac{1}{2\pi i} \int_{C_\alpha} \frac{F(z)}{z-p}\,dz + \frac{1}{2\pi i} \int_{a+ib}^{a-ib} \frac{F(z)}{z-p}\,dz$$

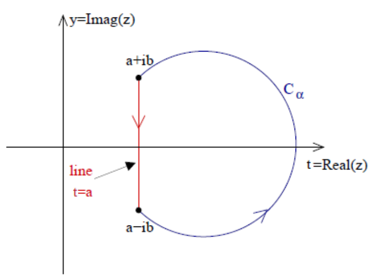

Figure 1: Integration curves for use with Cauchy's integral formula.

Now, since $F(z)$ is analytic to the right of $R(z) = a$, it is analytic and therefore continuous on C_α. This means that $F(z)$ is bounded on C_α ,

$|F(z)| \leq M$ on C_α for some constant M . Then by the ML (maximum-length) theorem ,

$$\left|\frac{1}{2\pi i}\int_{C_\alpha}\frac{F(z)}{z-p}\,dz\right| \leq \frac{M\pi b}{\min(|z-p|)}$$

$But\ |z-p| = |z-a-(p-a)| \geq |z-a| - |p-a| \geq b - |p-a|$
So ,

$$\left|\frac{1}{2\pi i}\int_{C_\alpha}\frac{F(z)}{z-p}\,dz\right| \leq \frac{M\pi b}{b-|p-a|}$$

$$\leq \frac{M\pi b}{1-|p-a|/b}$$

$$\longrightarrow 0 \text{ as b} \longrightarrow \infty$$

The last step uses $\mathrm{M} \to 0$ as $\mathrm{b} \to \infty$ which we can say since $F(z)$ is the Al-Tememe transform of the piecewise continuous $f\,(x)$ of exponential order. Thus

$$F\,(p) = \frac{1}{2\pi i}\int_{a+ib}^{a-ib}\frac{F\,(z)}{p-z}\,dz \quad \text{as b} \to \infty$$

We can re-write this as,

$$F(p) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{F\,(z)}{p-z}\,dz \quad \text{as b} \to \infty$$

Now we can invert the transform to recover $f(x)$,

$$f(x) = \mathcal{T}^{-1}[F(p)] = \mathcal{T}^{-1}\left\{\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{F(z)}{p-z}\,dz\right\}$$

$$= \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F\,(z)\,\mathcal{T}^{-1}\left[\frac{1}{p-z}\right]dz$$

But we know that $\mathcal{T}^{-1}\left[\frac{1}{p-z}\right] = x^{z-1}$

Thus ,

$$f(x) = \mathcal{T}^{-1}[F(p)] = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} x^{z-1}\,F(z)\,dz$$

Is the line integral that gives our inverse Al-Tememe transform. To compute the complex integral, we use the residue theorem. Let C_β be a unit circle centred at the origin (see Figure 2)

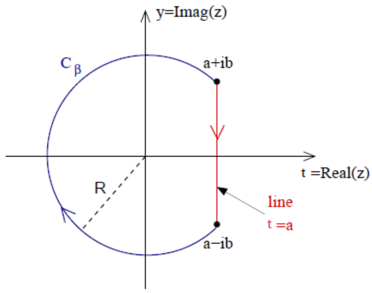

Figure 2: Integration curves for use in computing the inverse transform

$$\int_{a-ib}^{a+ib} x^{z-1}\,F(z)\,dz = \oint_{C} x^{z-1}\,F(z)\,dz + \int_{C_\beta} x^{z-1}\,F(z)\,dz$$

$$= 2\pi\,\mathrm{i}\sum_{j=1} K_j - \int_{C_\beta} x^{z-1}\,F(z)\,dz\,,$$

where the k_j are the residues of $x^{z-1}\,F(z)$ at the singularities of $F(z)$.

Now, by Jordan's lemma, if $|\,F(z)| \leq \mathrm{M}/\mathrm{R}^{\mathrm{k}}$ for some $\mathrm{k} > 0$ on C_β, then,

$$\lim_{R\to\infty}\int_{C_\beta} x^{z-1}\,F(z)\,dz = 0$$

Thus,

$$f(x) = \mathcal{T}^{-1}[F(p)] = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} x^{z-1}\,F(z)\,dz = \sum_{j=1} K_j$$

Remark: The inverse of Al-Tememe transform is not unique if

$$g(x) = \begin{cases} 1 & if \quad 1 < t < 3 \\ -7 & if \quad t = 3 \\ 1 & if \quad t > 3 \end{cases}$$

$$then\ \mathcal{T}(g(x)) = \frac{1}{p-1} \quad and \quad \mathcal{T}(x) = \frac{1}{p-1}$$

so both functions have the same Al-Tememe transform therefore,

$\frac{1}{p-1}$ has two inverse transform but the only continuous function with $\mathrm{Al-Tememe\ transform}$ $\frac{1}{p-1}$ is $f(x) = 1$.

Example(1): To find the inverse of Al-Tememe transform

$$F(p) = \frac{3}{(p-5)}$$

$We\ note\ that\ f(x)\ has\ a\ simple\ pole\ at\ p = 5\ and\ |f(x)| \leq \frac{M}{p-5}\ for\ all\ |p-5|$

suitably large.

$$Res(5) = \lim_{p \to 5} (p-5) x^{p-1} \frac{3}{(p-5)} = 3x^4$$

Example(2): To find the inverse of Al-Tememe transform

$$F(p) = \frac{3}{(p+7)^4}$$

We note that f(x) has a simple pole at $p = -7$ of order 4 nd $|f(x)| \leq \frac{M}{(p+7)^4}$,

for all $|p + 7|$ suitably large.

$$Res(-7) = \frac{1}{2} \lim_{p \to -7} \frac{d^3}{dp^3} \left\{ (p+7)^4 x^{p-1} \frac{6}{(p+7)^4} \right\}$$

$$= \frac{6}{2} \lim_{p \to -7} \frac{d^2}{dp^2} \{ lnx \ \ x^{p-1} \} = \frac{6}{2} \lim_{p \to 7} \frac{d}{dp} \{ (lnx)^2 \ \ x^{p-1} \}$$

$$= \frac{6}{2} \lim_{p \to -7} \{(lnx)^3 \ x^{p-1}\} = \frac{1}{2} x^{-7} (lnx)^3$$

Example(3): To find the inverse of Al-Tememe transform

$$F(p) = \frac{p+1}{p^2 - 2p + 5}$$

First we can write $F(p)$ as

$$\frac{p+1}{p^2-2p+5} = \frac{(p-1)+2}{(p-1)^2+4} = \frac{(p-1)}{(p-1)^2+4} + \frac{2}{(p-1)^2+4}$$
$$= F_1(p) + F_2(p)$$

$F_1(p) = \frac{(p-1)}{(p-1)^2+4}$ has two simple poles at $p = 1-2i$, $p = 1+2i$,

$$Res_1(1-2i) = \lim_{p \to 1-2i} \left\{ (p - (1-2i)) \, x^{p-1} \frac{(p-1)}{((p-1)-2i)\,((p-1)+2i)} \right\}$$
$$= \lim_{p \to 1-2i} \left\{ x^{p-1} \frac{(p-1)}{((p-1)-2i)} \right\} = \frac{x^{-2i}}{2i}$$

$$Res_1(1+2i) = \lim_{p \to 1+2i} \left\{ (p - (1+2i)) \, x^{p-1} \frac{(p-1)}{((p-1)-2i)\,((p-1)+2i)} \right\}$$
$$= \lim_{p \to 1+2i} \left\{ x^{p-1} \frac{(p-1)}{((p-1)+2i)} \right\} = \frac{x^{2i}}{2i}$$

$$f_1(x) = Res_1(1-2i) + Res_1(1+2i)$$
$$= \frac{x^{2i}}{2i} + \frac{x^{-2i}}{2i} = \frac{e^{2i\,lnx} + e^{-2i\,lnx}}{2i} = \cos(2lnx)$$

$F_2(p) = \frac{2}{(p-1)^2+4}$ has two simple poles at $p = 1-2i$, $p = 1+2i$

$$Res_2(1-2i) = \lim_{p \to 1-2i} \left\{ (p - (1-2i)) \, x^{p-1} \frac{2}{((p-1)-2i)\,((p-1)+2i)} \right\}$$
$$= \lim_{p \to 1-2i} \left\{ x^{p-1} \frac{2}{((p-1)-2i)} \right\} = \frac{x^{-2i}}{-2i}$$

$$Res_2(1+2i) = \lim_{p\to 1+2i}\left\{(p - (1+2i))\, x^{p-1}\frac{2}{\big((p-1)-2i\big)\big((p-1)+2i\big)}\right\}$$

$$= \lim_{p\to 1+2i}\left\{x^{p-1}\frac{2}{\big((p-1)+2i\big)}\right\} = \frac{x^{2i}}{2i}$$

$$f_2(x) = Res_2(1-2i) + Res_2(1+2i)$$

$$= \frac{x^{2i}}{2i} - \frac{x^{-2i}}{2i} = \frac{e^{2i\,lnx} - e^{-2i\,lnx}}{2i} = sin(2lnx)$$

$$f(x) = f_1(x) + f_2(x)$$

$$= sin(2lnx) + cos(2lnx)$$

Example(4): To find the inverse of Al-Tememe transform

$$F(p) = \frac{5}{(p-3)^2(p+1)}$$

We note that $f(x)$ has a three pole at $p = 3$ of order 2 and a polte at $p = -1$, and $|f(x)| \le \frac{M}{(p-3)^2(p+1)}$ for all $|p-3|$ suitably large.

$$Res(3) = \lim_{p\to 3}\frac{d}{dp}\left\{(p-3)^2 x^{p-1}\frac{5}{(p-3)^2(p+1)}\right\}$$

$$= \lim_{p\to 3}\left\{\frac{5\,x^{p-1}\,lnx\,(p+1) - 5x^{p-1}}{(p+1)^2}\right\} = \frac{5x^2 lnx}{4} - \frac{5x^2}{16}$$

$$Res(-1) = \lim_{p\to -1}\left\{(p+1)x^{p-1}\frac{5}{(p-3)^2(p+1)}\right\}$$

$$= \lim_{p\to -1}\left\{\frac{5\,x^{p-1}}{(p-3)^2}\right\} = \frac{5x^{-2}}{16}$$

$$f(x) = \frac{5}{4}x^2 lnx + \frac{5}{16}x^{-2} - \frac{5}{16}x^2$$

We solve the question by usual method (partial fraction)

$$\frac{5}{(p-3)^2(p+1)} = \frac{A}{p+1} + \frac{B}{p-3} + \frac{C}{(p-3)^2}$$

$$A + B = 0$$

$$-6A - 2B + C = 0$$

$$9A - 3B + C = 5$$

$Hence\ ,\ \ A = \frac{5}{16}\ ,\ \ A = \frac{-5}{16}\ \ ,\ \ C = \frac{5}{4}$

$\Longrightarrow f(x) = \frac{5}{16}x^{-2} - \frac{5}{16}x^{2} + \frac{5}{4}x^{2}lnx$

If $\mathcal{T}[f(x)]$

$= \frac{g(x)P(s)}{Q(s)}$ where g(x)is a function of x, P(s)and Q(s)are polynomials (having no common roots) of degree n and m, respectively, $m > n$ and

$Q(s)$has simple roots at $z_1, z_2, \dots, z_m$then $\mathcal{T}[f(x)]$ has a simple pole at each $p = z_k$ and writing

$$\mathcal{T}[f(x)] = F(p) = \frac{g(x)(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_o)}{(b_m s^m + b_{m-1}s^{m-1} + \cdots + b_o)}(a_n, b_m \neq 0)$$

$$= \frac{g(x)\left(a_n + \frac{a_{n-1}}{s} + \cdots + \frac{a_o}{s^n}\right)}{s^{m-n}\left(b_m + \frac{b_{m-1}}{s} + \cdots + \frac{b_o}{s^m}\right)}$$

It is enough to observe that for |s| suitably large ,

$$\left|a_n + \frac{a_{n-1}}{s} + \cdots + \frac{a_o}{s^n}\right| \le |a_n| + |a_{n-1}| + \cdots + |a_o| = c_1$$

$$\left|b_m + \frac{b_{m-1}}{s} + \cdots + \frac{b_o}{s^m}\right| \ge |b_m| - \frac{|b_{m-1}|}{|s|} - \cdots - \frac{|b_o|}{|s|^m} \ge \frac{|b_m|}{2} = c_2$$

So,

$|\mathcal{T}[f(x)]| \le \frac{c_1/c_2}{|s|^{m-n}}$. Then,

$$Res(z_k) = \frac{x^{z_k-1}P(z_k)}{Q'(z_k)}\ , k = 1,2, \dots, m$$

And ,

$$f(x) = \sum_{k=1}^{m}\frac{g(x)P(z_k)}{Q'(z_k)}x^{z_k-1}$$

Example(5): To find the inverse of

$$F(p) = \frac{p^2 + 7p - 1}{p^3 - 8p^2 + 19p - 12}$$

We note that $F(p)$ has simple poles at $z_1 = 1\ ,\ z_2 = 3\ ,\ z_3 = 4$ and $Q(p) = p^3 - 8p^2 + 19p - 12$

, $Q'(p) = 3p^2 - 16p + 19$,then ,

$$f(x) = \mathrm{Res}(z_1) + \mathrm{Res}(z_2) + \mathrm{Res}(z_3)$$
$$= x^{z_1-1}\frac{{z_1}^2 + 7z_1 - 1}{3{z_1}^2 - 16z_1 + 19} + x^{z_2-1}\frac{{z_2}^2 + 7z_2 - 1}{3{z_2}^2 - 16z_2 + 19} + x^{z_3-1}\frac{{z_3}^2 + 7z_3 - 1}{3{z_3}^2 - 16z_3 + 19}$$
$$= x^{1-1}\frac{(1)^2 + 7(1) - 1}{3(1)^2 - 16(1) + 19} + x^{3-1}\frac{(3)^2 + 7(3) - 1}{3(3)^2 - 16(3) + 19} + x^{4-1}\frac{(4)^2 + 7(4) - 1}{3(4)^2 - 16(4) + 19}$$
$$= \frac{43}{3}x^3 - \frac{29}{2}x^2 + \frac{7}{6}$$

Example(6): To find the inverse of

$$F(p) = \frac{3p^3 - 5p^2 + 4p - 2}{p^4 + p^3 - 19p^2 - 49p - 30}$$

We note that $F(p)$ has simple poles at $z_1 = -1$, $z_2 = -2$, $z_3 = -\ 3$, $z_4 = 5$ and
$Q(p) = p^4 + p^3 - 19p^2 - 49p - 30$, $Q'(p) = 4p^3 + 3p^2 - 38p - 49$, then ,

$$f(x) = Res(z_1) + Res(z_2) + Res(z_3) + Res(z_4)$$
$$= x^{-1-1}\frac{3(-1)^3 - 5(-1)^2 + 4(-1) - 2}{4(-1)^3 + 3(-1)^2 - 38(-1) - 49} + x^{-2-1}\frac{3(-2)^3 - 5(-2)^2 + 4(-2) - 2}{4(-2)^3 + 3(-2)^2 - 38(-2) - 49}$$
$$+ x^{-3-1}\frac{3(-3)^3 - 5(-3)^2 + 4(-3) - 2}{4(-3)^3 + 3(-3)^2 - 38(-3) - 49} + x^{5-1}\frac{3(5)^3 - 5(5)^2 + 4(5) - 2}{4(5)^3 + 3(5)^2 - 38(5) - 49}$$
$$= \frac{67}{84}x^4 + \frac{7}{6}x^{-2} - \frac{54}{7}x^{-3} + \frac{35}{4}x^{-4}$$

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