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On Linearly S-Quasi-Armendariz Rings

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Abstract

Let R be a ring, $(S,\leq)$ a strictly ordered monoid. Properties of the ring $[[R^{S,\leq}]]$ of generalized power series with coefficients in R and exponents in S are considered in this paper. We devoted to introduce and study linearly S-quasi-Armendariz ring, which is unify the notions of linearly S-Armendariz ring and S-quasi-Armendariz ring. It is shown that, if R is linearly S-quasi-Armendariz ring, U is a nonempty subset in R is a two-sided ideal of R, $A=l_R(U)$ for all $s\in S$, then R/A is linearly S-quasi-Armendariz. Also, we prove that, R is semiprime if and only if R is reduced linearly S-quasi-Armendariz. Moreover, a necessary and sufficient conditions are given for rings under which the classical left ring of quotients of R, is linearly S-quasi-Armendariz.

Mathematics Subject Classification: 16S36, 16U80, 20M25.

Key words: generalized power series ring, linearly S-quasi-Armendariz ring, semicommutative ring.

1. INTRODUCTION

All rings considered here are associative with identity. We will write monoids multiplicatively unless otherwise indicated. If R is a ring and X is a nonempty subset of R, then the left (right) annihilator of X in R is denoted by $l_R(X)(r_R(X))$ Any concept and notation not defined here can be found in Elliott and Ribenboim [6] and Ribenboim [20].

Rege and Chhawchharia [14] introduced the notion of an Armendariz ring. They defined a ring R to be an Armendariz ring if whenever polynomials $f(x)=a_0+a_1x+\ldots+a_nx^n, g(x)=b_0+b_1x+\ldots+b_mx^m \in R[x]$ satisfy $f(x)g(x)=0$, then $a_ib_j=0$ for each i,j. (The converse is always true.) The name "Armendariz ring" was chosen because Armendariz [5, Lemma 1] had noted that a reduced ring satisfies this condition. Reduced rings (i.e., rings with no nonzero nilpotent elements). Some properties of Armendariz rings have been studied in E. P. Armendariz [5], Anderson and Camillo [3], Kim and Lee [16], Huh, Lee and Smoktunowicz [2], and Lee and Wong [21].

By Kim et al. in [15]. A ring R is said to be power-serieswise Armendariz if whenever power series $f(x)=a_0+a_1x+\ldots+a_nx^n, g(x)=b_0+b_1x+\ldots+b_mx^m \in R[[x]]$ satisfy
$f(x)g(x)=0$ then $a_ib_j=0$ for all i,j. Armendariz rings were generalized to quasi-Armendariz rings by Hirano [23]. A ring R is called quasi-Armendariz provided that $a_iRb_j=0$ for all i,j whenever $f(x)=a_0+a_1x+\ldots+a_nx^n, g(x)=b_0+b_1x+\ldots+b_mx^m \in R[x]$ satisfy $f(x)R[x]g(x)=0$.

Let $(S,\leq)$ be an ordered set. Recall that $(S,\leq)$ is artinian if every strictly decreasing sequence of elements of S is finite, and that $(S,\leq)$ is narrow if every subset of pairwise orderincomparable elements of S is finite. Thus, $(S,\leq)$ is artinian and narrow if and only if every nonempty subset of S has at least one but only a finite number of minimal elements. Let S be a commutative monoid. Unless stated otherwise, the operation of S will be denoted additively, and the neutral element by 0. The following definition is due to Elliott and Ribenboim [6].

Let $(S,\leq)$ is a strictly ordered monoid (that is, $(S,\leq)$ is an ordered monoid satisfying the condition that, if $s,s',t\in S$ and $s<s'$, then $s+t<s'+t$, and R a ring. Let $[[R^{S,\leq}]]$ be the set of all maps $f:S\to R$ such that supp(f) $=\{s\in S \mid f(s)\neq 0\}$. is artinian and narrow. With pointwise addition, $[[R^{S,\leq}]]$ is an abelian additive group. For every $s\in S$ and $f,g\in[[R^{S,\leq}]]$, let $X_S(f,g)=\{(u,v)\in S\times S \mid u+v=s, f(u)\neq 0, g(v)\neq 0\}$. It follows from Ribenboim [20, 4.1] that $X_S(f,g)$ is finite. This fact allows one to define the operation of convolution:

$$(fg)(s)=\sum_{(u,v)\notin X_S(f,g)} f(u)g(v).$$

Clearly, supp(fg) $\subseteq$ supp(f) + supp(g), thus by Ribenboim [18, 3.4] supp(fg) is artinian and narrow, hence $f,g\in[[R^{S,\leq}]]$. With this operation, and pointwise addition, $[[R^{S,\leq}]]$ becomes an associative ring, with identity element e, namely $e(0)=1, e(s)=0$ for every $0\neq s\in S$. Which is called the ring of generalized power series with coefficients in R and exponents in S. Many examples and results of rings of generalized power series are given in Ribenboim ([17]−[20]), Elliott and Ribenboim [6] and Varadarajan ([12], [13]). For example, if $S=N\bigcup\{0\}$ and $\leq$ is the usual order, then $[[R^{N\bigcup\{0\},\leq}]]\cong R[[x]]$, the usual ring of power series. If S is a commutative monoid and $\leq$ is the trivial order, then $[[R^{S,\leq}]]\cong R[S],$, the monoid ring of S over R.

Further examples are given in Ribenboim [18]. To any $r \in R$ and $s \in S$, we associate the maps $c_r, c_s \in [[R^{S,\leq}]]$ defined by

$$c_r(x) = \begin{cases} r, x = 0 \\ 0, otherwise, \end{cases} \qquad e_s(x) = \begin{cases} 1, x = s \\ 0, otherwise, \end{cases}$$

It is clear that $r \to c_r$ is a ring embedding of R into $[[R^{S,\leq}]]$, $s \leftarrow e_s$, is a monoid embedding of S into the multiplicative monoid of the ring $[[R^{S,\leq}]]$, and $c_r e_s = e_s c_r$. Recall that a monoid S is torsion-free if the following property holds: If $s, t \in S$, if k is an integer, $k \geq 1$ and $ks = kt$, then $s = t$.

If R is a ring and S is a strictly ordered monoid, then the ring R is called a generalized Armendariz ring if for each $f, g \in [[R^{S,\leq}]]$ such that $fg = 0$ implies that $f(u)g(v) = 0$ for each $u \in$ supp(f) and $v \in$ supp(g). In Liu. [24] called such ring S-Armendariz ring. Ali and Elshokry in [4], said that, if R is a ring, S be a torsion-free and cancellative monoid and $\leq a$ strict order on S, then the ring R is called a generalized quasi-Armendariz ring (S- quasi-Armendariz), if for each $f, g \in [[R^{S,\leq}]]$ such that $f[[R^{S,\leq}]]g = 0$, then $f(u)Rg(v) = 0$ for each $u, v \in S$.

In this paper, we introduce the new concept of linearly S-quasi-Armendariz which is unify the notions of S-quasi-Armendariz ring and linearly S-Armendariz (they defined that a ring R is linearly S-Armendariz, if whenever $f, g \in [[R^{S,\leq}]]$ such that $fg = 0$, then $a_i b_j = 0$ for all $a_0, a_1, b_0, b_1 \in R$, such that $a_0 b_0 = a_0 b_1 = a_1 b_0 = a_1 b_1 = 0$, where $f = c_{a_0} + c_{a_1} e_s, g = c_{b_0} + c_{b_1} e_s$. It is shown that, (‡) If R is linearly S-quasi-Armendariz ring, U is a nonempty subset in R is a two-sided ideal of R, $A = l_R(U)$ for all $s \in S$, then R/A is linearly S-quasi-Armendariz. (‡) For a two-sided ideal I of R, if R/I is a linearly S-quasi-Armendariz ring and I is a semiprime ring without identity, then R is linearly S-quasi-Armendariz. Moreover, (‡) Under a necessary and sufficient conditions, if R is a linearly S-quasi-Armendariz, then Q is a linearly S-quasi-Armendariz, where Q is the classical left ring of quotients of R. Consequently, some results of a linearly S-quasi-Armendariz are given.

Clark defined quasi-Baer rings in [22]. A ring R is called quasi-Baer if the left annihilator of every left ideal of R is generated by an idempotent. Birkenmeier, Kim and Park in [8] introduced the concept of principally quasi-Baer rings. A ring R is called left principally quasi-Baer (or simply left p.q.-Baer) if the left annihilator of a principal left ideal of R is generated by an idempotent. Similarly, right p.q.-Baer rings can be defined. A ring is called p.q.-Baer if it is both right and left p.q.-Baer. Observe that biregular rings and quasi-Baer rings are p.q.-Baer. For more details and examples of left p.q.-Baer rings, (see [7] and [8]). A ring R is called a right (resp., left) PP-ring if every principal right (resp., left) ideal is projective (equivalently, if the right (resp., left) annihilator of an element of R is generated (as a right (resp., left) ideal) by an idempotent of R).

2. Linearly S-quasi-Armendariz rings

Definition 2.1. Let R be a ring, $(S,\leq)$ a strictly totally ordered monoid. We say that
R is linearly S-quasi-Armendariz, if for all $s \in S \setminus \{1\}$ and
$a_0, a_1, b_0, b_1 \in R$, whenever
$(c_{a_0} + c_{a_1}e_s)[[R^{S,\leq}]](c_{b_0} + c_{b_1}e_s) = 0$, then $a_i R b_j = 0$, for all
$0 \leq i \leq 1, 0 \leq j \leq 1$.

Definition 2.2. Let S be a ring, $(S,\leq)$ a strictly totally ordered monoid. We say that R is linearly S-Armendariz, if for all $s\in S\setminus\{1\}$ whenever $(c_{a_0}+c_{a_1}e_s)(c_{b_0}+c_{b_1}e_s)=0$ in $[[R^{S,\leq}]]$, then $a_0b_0=a_0b_1=a_1b_0=a_1b_1=0$ in R.

It can be easily checked that both S-quasi-Armendariz rings and linearly S-Armendariz rings are linearly S-quasi-Armendariz. But there exist linearly S-quasi-Armendariz rings which are not linearly S-Armendariz e.g., $Mat_2(R)$ over a linearly S-quasi-Armendariz ring R is linearly S-quasi-Armendariz by [1, Theorem 2.3], but $\mathrm{Mat}_2(R)$ is not linearly S-Armendariz by [14] (or [16, Example 1]), even in the case where R is commutative and $[[R^{S,\leq}]]=R[x]$. Also, the construction in [21, Example 3.2] shows that there exist commutative linearly S-quasi-Armendariz rings which are not S-quasi-Armendariz, even in the case, S be the additive monoid $\mathbb{N}\cup\{0\}$, with the trivial order, R be a ring. Then R is an S-quasi-Armendariz ring if and only if R is a quasi-Armendariz ring in the usual sense. This is so because in this case the generalized power series ring $[[R^{S,\leq}]]$ is isomorphic to the ordinary polynomial ring $R[x]$.

The next Lemma appeared in [9, Lemma 1.2].

Lemma 2.3. For any ring R the following are equivalent:

(1) For each element $a\in R$, a^r is an ideal of R, where $a^r=\{b\in R: ab=0\}$.

(2) Any annihilator right ideal of R is an ideal of R.

(3) Any annihilator left ideal of R is an ideal of R.

(4) For any $a,b\in R, ab=0$ implies $aRb=0$.

Every reduced ring (i.e., if there exists no nonzero nilpotent elements) is semicommutative but the converse does not hold in general. There exists a linearly S-quasi-

Armendariz ring which is not semicommutative [2, Example 14], even in the case where R is commutative and $[[R^{S,\leq}]] = R[x]$, and commutative (hence semicommutative) rings need not to be linearly S-quasi-Armendariz. Here we have the following.

Proposition 2.4. Let $[[R^{S,\leq}]]$ over a ring R be semicommutative, $(S,\leq)$ a strictly totally ordered monoid. If R is (linearly) S-quasi-Armendariz, then R is (linearly) S-Armendariz.

Proof. Since the two cases have the same argument, we only give the proof of S-Armendariz case. Assume that the generalized power series ring $[[R^{S,\leq}]]$ over R is S-quasi-Armendariz and semicommutative. Let $f, g \in [[R^{S,\leq}]]$ such that $fg = 0$. Then we get $f[[R^{S,\leq}]]g = 0$ and so $f(u)Rg(v) = 0$, for all $u, v \in S$. Thus, $f(u)g(v) = 0$ for all $u, v \in S$, and therefore R is S-Armendariz.

The following result appeared in [24].

Lemma 2.5. Let R be a ring, $(S,\leq)$ a strictly ordered monoid. Then $[[R^{S,\leq}]]$ is reduced if and only if R is reduced.

Lemma 2.6. [25] Let R be a ring, $(S,\leq)$ a strictly totally ordered monoid. Then R is a semiprime ring if and only if $[[R^{S,\leq}]]$ is a semiprime ring.

Since any reduced ring is a semiprime. Here we have.

Theorem 2.7. For a ring R, $(S,\leq)$ a strictly totally ordered monoid. We consider the following conditions:

(1) R is semiprime.

(2) $[[R^{S,\leq}]]$ is linearly S-quasi-Armendariz.

(3) R is linearly S-quasi-Armendariz.

Then $(1) \Rightarrow (2) \Rightarrow (3)$.

Proof. Observe that R is a semiprime ring if and only if so is $[\![R^{S,\leq}]\!]$ by Lemma 2.6.

$(1) \Rightarrow (2)$ Suppose that R is semiprime. Then $[\![R^{S,\leq}]\!]$ is S-quasi-Armendariz by [4, Proposition 2.18], and so $[\![R^{S,\leq}]\!]$ is linearly S-quasi-Armendariz.

$(2) \Rightarrow (3)$ Assume that $[\![R^{S,\leq}]\!]$ is linearly S-quasi-Armendariz. Let $f = c_{a_0} + c_{a_1}e_s$ $g = c_{b_0} + c_{b_1}e_s$ $\in [\![R^{S,\leq}]\!]$ such that $(c_{a_0} + c_{a_1}e_s)[[R^{S,\leq}]](c_{b_0} + c_{b_1}e_s) = 0$. Thus, we have the following equations, for any $r \in R$

$$a_0rb_0 = 0, a_0rb_1 + a_1rb_0 = 0, a_1rb_1 = 0(*).$$

Let $h = c_{a_0} + c_{a_1}e_s, l = c_{b_0} + c_{b_1}e_s \in [[R^{S,\leq}]]$, for any $t \in S$. We claim that $h[[R^{S,\leq}]]l = 0$. For any $d \in R$ and any $t \in S$. By $(*)$, we have $hc_d l = (a_0db_0) + (a_0db_1 + a_1db_0)e_t + (a_1db_1)e_t = 0$, because $e_se_t = e_te_s$. Since $[\![R^{S,\leq}]\!]$ is linearly S-quasi-Armendariz, we have $c_{a_i}[\![R^{S,\leq}]\!]c_{b_j} = 0$, for all i, j. In particular, $a_iRb_j = 0$, for all i, j, and therefore R is linearly S-quasi-Armendariz.

Corollary 2.8. ([1, Theorem 2,7]). For a ring R, we consider the following conditions:

(1) R is semiprime.

(2) $R[x]$ is linearly quasi-Armendariz.

(3) R is linearly quasi-Armendariz.

Then $(1) \Rightarrow (2) \Rightarrow (3)$.

Corollary 2.9. Let $(S, \leq)$ be a strictly totally ordered monoid. If R is a semiprime, then R is linearly S-quasi-Armendariz.

Proof. By [4, Proposition 2.18], R is S-quasi-Armendariz. Thus, R is linearly S-quasi- Armendariz.

Proposition 2.10. Let $(S,\leq)$ be a strictly ordered monoid and e be a central idempotent of a ring R. Then the following statements are equivalent:

(1) R is linearly S-quasi-Armendariz;

(2) eR and $(1-e)R$ are linearly S-quasi-Armendariz.

Proof. $(1)\Rightarrow(2)$. Suppose that R is linearly S-quasi-Armendariz. Let $c_{a_0}+c_{a_1}e_s$ and $c_{b_0}+c_{b_1}e_s\in[[(eR)^{S,\leq}]]$, such that $(c_{a_0}+c_{a_1}e_s)[[(eR)^{S,\leq}]](c_{b_0}+c_{b_1}e_s)=0$. Note that $(c_{a_0}+c_{a_1}e_s)c_e=c_{a_0}+c_{a_1}e_s$ and $(c_{a_0}+c_{a_1}e_s)=c_{b_0}+c_{b_1}e_s$. For any $r\in R$, $(c_{a_0}+c_{a_1}e_s)c_r(c_{b_0}+c_{b_1}e_s)$ $=(c_{a_0}+c_{a_1}e_s)(c_{er})(c_{b_0}+c_{b_1}e_s)=0$, and so $(c_{a_0}+c_{a_1}e_s)[[R^{S,\leq}]](c_{b_0}+c_{b_1}e_s)=0$. Since R is linearly S-quasi-Armendariz, $a_0Rb_0=a_0Rb_1=a_1Rb_0=a_1Rb_1=0$. Since e is central $a_0(eR)b_0=0, a_0(eR)b_1=0, a_1(eR)b_0=0$ and $a_1(eR)b_1=0$.

Therefore, eR is linearly S-quasi-Armendariz. Similarly, we can show that $(1-e)R$ is linearly S-quasi-Armendariz.

$(2)\Rightarrow(1)$. Assume that both eR and $(1-e)R$ are linearly S-quasi-Armendariz. Let $c_{a_0}+c_{a_1}e_s$ and $c_{b_0}+c_{b_1}e_s\in[[R^{S,\leq}]]$ be such that $(c_{a_0}+c_{a_1}e_s)[[R^{S,\leq}]](c_{b_0}+c_{b_1}e_s)=0$.

We will show that $a_0Rb_0=0, a_0Rb_1=0, a_1Rb_0=0$ and $a_1Rb_1=0$. For any $r\in R$, $c_e(c_{a_0}+c_{a_1}e_s)(c_{er})c_e(c_{b_0}+c_{b_1}e_s)=c_e((c_{a_0}+c_{a_1}e_s)c_r(c_{b_0}+c_{b_1}e_s))=0$ and $c_{1-e}(c_{a_0}+c_{a_1}e_s)c_{1-e}c_r(c_{1-e}(c_{b_0}+c_{b_1}e_s))=0$, so $c_e(c_{a_0}+c_{a_1}e_s)[[(eR)^{S,\leq}]]c_e(c_{b_0}+c_{b_1}e_s)=0$ and $c_{1-e}(c_{a_0}+c_{a_1}e_s)[[(1-eR)^{S,\leq}]]c_{1-e}(c_{b_0}+c_{b_1}e_s)=0$

Since eR and $(1-e)R$ are linearly S-quasi-Armendariz, we have $e(a_0Rb_0)=0, e(a_0Rb_1)=0, e(a_1Rb_0)=0, e(a_1Rb_1)=0$ and $(1-e)(a_0Rb_0)=0,(1-e)(a_0Rb_1)=0$ $,(1-e)(a_1Rb_0)=0,(1-e)(a_1Rb_1)=0$ and hence $a_0Rb_0=e(a_0Rb_0)+(1-e)(a_0Rb_0)=0$,

$a_0Rb_1=e(a_0Rb_1)+(1-e)(a_0Rb_1)=0$,

$a_1Rb_0=e(a_1Rb_0)+(1-e)(a_1Rb_0)=0,$

$a_1Rb_1=e(a_1Rb_1)+(1-e)(a_1Rb_1)=0,$. Therefore, R is linearly S-quasi-Armendariz.

Theorem 2.11. Let R be a ring, $(S,\leq)$ a strictly totally ordered monoid. Then the following conditions are equivalent:

(1) R is semiprime;

(2) R is reduced linearly S-quasi-Armendariz.

Proof. $(1)\Rightarrow(2)$ Is trivial.

$(2)\Rightarrow(1)$ Let R be a reduced linearly S-quasi-Armendariz. In particular for any $C_a\in[[R^{S,\leq}]]$ be such that $C_a[[R^{S,\leq}]]C_a=0$, then $aRa=0$. Thus, by reduced $(aR)^2=0$.

Therefore $a=0$.

Corollary 2.12. Let R be a ring, $(S,\leq)$ a strictly ordered monoid. If R is reduced ring, the R is linearly S-quasi-Armendariz.

Theorem 2.13. Let R be a ring, $(S,\leq)$ a strictly ordered monoid.

(1) If R is linearly S-quasi-Armendariz ring, U is a nonempty subset in R is a two-sided ideal of R, $A=l_R(U)$ for all $s\in S$. Then R/A is linearly S-quasi-Armendariz.

(2) For a two-sided ideal I of R, if R/I is a linearly S-quasi-Armendariz ring and I is a semiprime ring without identity, then R is linearly S-quasi-Armendariz.

Proof. (1) Assume that $A = r_R(U)$ is a two sided of linearly S-quasi-Armendariz ring R for $\phi \neq U \subseteq R$. Let $\overline{a} = a + A$ for $a \in R$. Suppose $c_{\overline{a}_0} + c_{\overline{a}_1} e_s$ and $c_{\overline{b}_0} + c_{\overline{b}_1} e_s \in [[\overline{R}^{S,\leq}]]$ with $(c_{\overline{a}_0} + c_{\overline{a}_1} e_s)[[\overline{R}^{S,\leq}]](c_{\overline{b}_0} + c_{\overline{b}_1} e_s) = \overline{0}$. We claim that

$\overline{a}_0 (R/A)\overline{b}_0 = 0, \overline{a}_0 (R/A)\overline{b}_1 = 0$

$\overline{a}_1 (R/A)\overline{b}_0 = 0$ and $\overline{a}_1 (R/A)\overline{b}_1 = 0$.

From $(c_{\overline{a}_0} + c_{\overline{a}_1} e_s)[[(R/A)^{S,\leq}]](c_{\overline{b}_0} + c_{\overline{b}_1} e_s) = \overline{0}$, we get

$(c_{\overline{a}_0} + c_{\overline{a}_1} e_s) c_{\overline{r}} (c_{\overline{b}_0} + c_{\overline{b}_1} e_s) = \overline{0}$ for any $\overline{r} \in R/A$.

Hence $a_0 r b_0, a_0 r b_1 + a_1 r b_0, a_1 r b_1 \in A$,

and so $t a_0 r b_0 = 0, t(a_0 r b_1 + a_1 r b_0) = 0, t a_1 r b_1 = 0$ for any $r \in R$ and $t \in U$. Thus,

$c_t (c_{a_0} + c_{a_1} e_s)[[R^{S,\leq}]](c_{b_0} + c_{b_0} e_s) = 0$. Since R is linearly S-quasi-Armendariz, we have

$t(a_0 r b_0) = 0, t(a_0 r b_1) = 0, t(a_1 r b_0) = 0$ and $t(a_1 r b_1) = 0$ for any $t \in U$, and hence

$a_0 r b_0 \subseteq A, a_0 r b_1 \subseteq A, a_1 r b_0 \subseteq A$ and $a_1 r b_1 \subseteq A$. Thus,

$\overline{a}_0 (R/A)\overline{b}_0 = 0, \overline{a}_0 (R/A)\overline{b}_1 = 0, \overline{a}_1 (R/A)\overline{b}_0 = 0$ and $\overline{a}_1 (R/A)\overline{b}_1 = 0$ and

therefore R/A is linearly S-quasi-Armendariz.

(2) Let $c_{a_0} + c_{a_1} e_s$ and $c_{b_0} + c_{b_1} e_s \in [[R^{S,\leq}]]$, such that $(c_{a_0} + c_{a_1} e_s)[[R^{S,\leq}]](c_{b_0} + c_{b_1} e_s) = 0$. Then we have $a_0 r b_0 = 0, a_0 r b_1 + a_1 r b_0 = 0$ and $a_1 r b_1 = 0$ for any $r \in R$, thus $a_0 R b_0 = 0$ and $a_1 R b_1 = 0$. We claim that $a_0 R b_1 = 0$. Assume $a_0 R b_1 \neq 0$. Note that $(b_0 I a_0 R)^2 = 0$

implies $b_0Ia_0R=0$ and so $b_0Ia_0=0$ since $b_0Ia_0R\subseteq I$ and I is semiprime. Since R/I is linearly S-quasi-Armendariz, we get $a_0Rb_0\subseteq I, a_0Rb_1\subseteq I, a_1Rb_0\subseteq I$ and $a_1Rb_1\subseteq I$.
Then

$$(a_1Rb_0)(Ra_0Rb_1)^2=(a_1R)(b_0)(Ra_0Rb_1Ra_0)Rb_1\subseteq a_1R(b_0)Ia_0(Rb_1)=0$$

From $a_0rb_1+a_1rb_0=0$ for any $r\in R$, we have

$$0=(a_0rb_1+a_1rb_0)(ua_0tb_1)^2=a_0rb_1(ua_0tb_1)^2$$

for any $r,u,t\in R$ and thus $(Ra_0Rb_1)^3=0$. Since $Ra_0Rb_1\subseteq I$ and I is semiprime,
$Ra_0Rb_1=0$ and so $a_0Rb_1=0$, a contradiction. Hence, $a_0Rb_0=0, a_0Rb_1=0, a_1Rb_0=0$ and $a_1Rb_1=0$ and therefore R is linearly S-quasi-Armendariz.

Remark 2.14. Let $R=Z_2\oplus Z_2$. It can be easily checked that R is a linearly S-quasi-Armendariz and semicommutative ring, and hence R/A is linearly S-quasi-Armendariz ring for the one-sided annihilator A of a nonempty subset in R by Theorem 2.13(1). Moreover, $R/I\cong Z_2$ is a linearly S-quasi-Armendariz ring for a semiprime ideal $I=\{0\}\oplus R$ of R, even in the case where R is commutative and $[[R^{S,\leq}]]=R[x]$.

Corollary 2.15. Let R be a ring, $(S,\leq)$ a strictly ordered monoid.

(1) If a ring R is semicommutative and linearly S-quasi-Armendariz, then R/A is linearly S-quasi-Armendariz, where $A=l_R(U)$ and U is a nonempty subset in R.

(2) If a ring R is linearly S-quasi-Armendariz and satisfies any one of the following conditions, then R/A is linearly S-quasi-Armendariz:

- R is an abelian Baer ring and A is the one-sided annihilator of a nonempty subset in R.
- R is a quasi-Baer ring and A is the right annihilator of a right ideal in R.
- R is an abelian right (resp., left) p.p.-ring and A is the right (resp., left) annihilator of an element in R.
- R is a right (resp., left) p.q.-Baer ring and A is the right (resp., left) annihilator of a principal right (resp., left) ideal in R.

Proof. (1) By Lemma 2.3, a ring R is semicommutative ring if and only if any one-sided annihilator over R is a two-sided ideal of R, and thus R/A is linearly S-quasi-Armendariz by Theorem 2.13.

(2) If R is abelian or A is the right (resp., left) annihilator of a right (resp., left) ideal in R, then A is a two-sided ideal of R. Thus, R/A is linearly S-quasi-Armendariz by Theorem 2.13.

One can find the next definition in [11].

Definition 2.16. Let $(S,\leq)$ be an ordered monoid. We say that $(S,\leq)$ is an artinian narrow unique product monoid (or an a.n.u.p. monoid, or simply a.n.u.p.) if for every two artinian and narrow subsets X and Y of S there exists a u.p. element in the product XY. We say that $(S,\leq)$ is quasitotally ordered (and that $\leq$ is a quasitotal order on S) if $\leq$ can be refined to an order $\preceq$ with respect to which S is a strictly totally ordered monoid. For any ordered monoid $(S,\leq)$, the following chain of implications holds: S is commutative, torsion-free, and cancellative

$$\Downarrow$$

$(S,\leq)$ is quasitotally ordered

$\Downarrow$

$(S,\leq)$ is a.n.u.p.

$\Downarrow$

S is u.p.

The converse of the bottom implication holds if $\leq$ is the trivial order. For more details, examples, and interrelationships between these and other conditions on ordered monoids, we refer the reader to [10].

Let R be a semiprime left Goldie ring, and let C denote the set of regular elements of R (that is, elements that are neither left nor right zero-divisors). If $Q = Q_{cl}^{l}$ is the classical left ring of quotients of R. Then we have for a monoid S. The following result generalizes [1, Theorem 2.16].

Theorem 2.17. Let R be a semiprime left Goldie ring, $(S,\leq)$ a nontrivial strictly ordered a.n.u.p. monoid. Let $Q = Q_{cl}^{l}$ denote the classical left ring of quotients of R. Then the following conditions are equivalent:

(1) R is S-quasi-Armendariz;

(2) R is linearly S-quasi-Armendariz;

(3) Q is S-quasi-Armendariz;

(4) Q is linearly S-quasi-Armendariz.

Proof. $(1) \Rightarrow (2)$ Trivial.

$(2) \Rightarrow (4)$ We have to show that for any $p_0, p_1, q_0, q_1 \in Q$ and $s \in S \setminus \{1\}$, if $(c_{p_0} + c_{p_1} e_s)[[Q^{S,\leq}]](c_{q_0} + c_{q_1} e_s) = 0$, then $p_0 r q_1 = p_1 r q_0 = 0$. (‡)

Now, there exist $a_0, a_1, b_0, b_1, u \in R$ such that u is regular and $p_i = u^{-1} a_i, q_i = u^{-1} b_i$ for $i = 1,2$. Furthermore, for some $d_0, d_1, v \in R$ with v regular, we can write $a_0 u^{-1} = v^{-1} d_0$ and $a_1 u^{-1} = v^{-1} d_1$. Now it is easy to see that in $[[R^{S,\leq}]]$ we have

$(c_{d_0} + c_{d_1} e_s)[[R^{S,\leq}]](c_{b_0} + c_{b_1} e_s) = 0$, Since R is linearly S-quasi-Armendariz, we obtain $d_0 r b_1 = d_1 r b_0 = 0$. Now $p_0 r q_1 = p_1 r q_0 = 0$ follows easily, proving (‡).

$(3) \Leftrightarrow (4)$ Trivial.

The following is obtained by applying the method in the proof of Theorem 2.17.

Proposition 2.18. Let R be a semiprime left Goldie ring, $(S,\leq)$ a nontrivial strictly ordered a.n.u.p. monoid. Let Δ be a multiplicatively closed subset of a ring R consisting of central regular elements. Then R is linearly S-quasi-Armendariz if and only if $\Delta^{-1}R$ is linearly S-quasi-Armendariz.

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