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# On Two-Dimensional Motion of Incompressible Variable Viscosity Fluids with Moderate Peclet Number Via von-Mises Coordinates

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**Abstract**

*The problem of plane steady motion with moderate Peclet number of incompressible and variable viscosity fluid is addressed to announce a class of exact solutions in von-Mises coordinates. The class is characterized by an equation relating a differentiable function* $f(x)$ *and a stream function* $\psi$ *. The successive transformation technique is applied for exact solutions. The study finds* $f(x)$ *arbitrary in one case and a specific value in other case. In both the cases, this discourse shows an infinite set of new exact solutions for moderate Peclet number.*

**Key words:** Variable viscosity fluids, Navier-Stokes equations, Exact solutions to incompressible fluids, Martin's system, von-Mises coordinates, Moderate Peclet number.

## 1 Introduction

The momentum of moving fluid element in presence or absence of body force is given by Navier-Stokes equations (NSE) but its mathematical structure is complex. A variety of techniques/methods including references given in [1-6] finds some exact solutions of NSE. Moreover recently Mushtaq A. *et. al.* [7], applied a new technique for exact solution of equations for flow of fluids possessing variable viscosity. Mushtaq A. [10, 14], and Mushtaq A. *et. al.* [11-13] applied successive transformation technique to find exact solutions for flow of incompressible fluids of variable viscosity with body force. The successive transformation technique used in [10-14] was to first transform the basic equations in curvilinear coordinates $(\varphi,\psi)$. The resulting equations were then transformed defining the curvilinear coordinate $\psi$ as streamline, selecting the curvilinear coordinate $\varphi$ in some specific way and assuming angle $\alpha$ between these coordinate curves. In fact the idea of this transformation is a loan from Martin [15] where the curvilinear coordinate $\psi$ is defined as streamline and the curvilinear coordinate $\varphi$ is left arbitrary therefore the coordinates $(\varphi,\psi)$ will be called here as "Martin's coordinates".

The arbitrary coordinate lines $\varphi = \text{constant}$ of Martin's coordinates system $(\varphi,\psi)$ can be set in many ways. For example, the technique of [10-14] was to retransform the basic equations taking angle $\alpha$ between Martin's coordinates taking the coordinate $\varphi$ along the radial direction. Another example is the von-Mises coordinates $(x,\psi)$ here the coordinate lines $\varphi = \text{constant}$ of Martin's coordinates is taken along $x-axis$ thus the function $\varphi = x$ and stream function $\psi$ of Martin's coordinates as independent variables instead of $y$ and $x$ [16].

This communication first transforms the basic non-dimensional flows equations from Cartesian space $(x,y)$ to Martin's coordinates then to von-Mises coordinates $(x,\psi)$. It further characterizes the streamlines of the class of flows under consideration by

$$\frac{y-f(x)-b}{a} = \text{const.} \tag{1}$$

Where $f(x)$ is a differentiable function and $a \neq 0$, $b$ are constants. The equation (1) implies

$$y = f(x) + \nu(\psi) \tag{2}$$

where $\nu(\psi) = a\psi + b$.

The paper is organized in sections: Section (2) gives basic non-dimensional flow equations and transforms them into Martin system $(\varphi,\psi)$. Section (3) retransforms the basic equations to von-Mises coordinates. The exact solutions to the problem are given in section 4. Conclusions are given at the end.

## 2. Basic non-dimensional equations in Martin's coordinates

Consider the following non-dimensional form of the fluid flow model for steady plane motion of incompressible fluid of variable viscosity with constant thermal conductivity with no heat addition [7]

$$u_x + v_y = 0 \tag{3}$$

$$u\,u_x + v\,u_y = -\,p_x + \frac{1}{\mathrm{R_e}}\,[(2\mu u_x)_x + \{\mu(u_y + v_x)\}_y] \tag{4}$$

$$u\,v_x + v\,v_y = -\,p_y + \frac{1}{\mathrm{R_e}}\,[(2\mu v_y)_y + \{\mu(u_y + v_x)\}_x] \tag{5}$$

$$u\,T_x + v\,T_y = \frac{1}{\mathrm{R_e\,P_r}}\,(T_{xx} + T_{yy}) + \frac{\mathrm{E_c}}{\mathrm{R_e}}\,[2\mu(u_x{}^2 + v_y{}^2) + \mu(u_y + v_x)^2] \tag{6}$$

Where $u$, $v$ are the components of velocity vector $\mathbf{q} = (u,v)$, $T$ is temperature, $p$ pressure, $\mu$ the viscosity. These five quantities are function of $x$ and $y$. The Reynolds number, the Ecart number and the Prandtl numbers are denoted by $R_e$, $E_c$ and $P_r$ are respectively. The product of $R_e$ and $P_r$ is Peclet number $P_{e'}$. The solution of these equations for very large and very small $P_{e'}$ can be found where as finding solutions for moderate $P_{e'}$ is challenging [17-22].

A stream function $\psi(x, y)$ satisfies equation (3) such that

$$u = \psi_y \qquad v = -\psi_x \tag{7}$$

The equations (4-6) in manageable form by introducing the vorticity function $\Omega$ and the total energy function $L$ defined by

$$\Omega = v_x - u_y \tag{8}$$

$$L = p + \frac{1}{2}(u^2 + v^2) - \frac{1}{\mathrm{R_e}}(2\mu u_x) \tag{9}$$

are

$$-v\,\Omega = -L_x + \frac{1}{\mathrm{R_e}} A_y \tag{10}$$

$$u\,\Omega = -L_y - \frac{1}{R_e} B_y + \frac{1}{\mathrm{R_e}} A_x \tag{11}$$

$$u\,T_x + v\,T_y = \frac{1}{P_{e'}}(T_{xx} + T_{yy}) + \frac{\mathrm{E_c}}{\mathrm{R_e}} \frac{1}{4\mu}\left(B^2 + 4A^2\right) \tag{12}$$

where

$$A = \mu(u_y + v_x) \quad \text{and} \quad B = 4\mu\, u_x \tag{13}$$

Consider a curvilinear coordinate system $(\varphi,\psi)$ in the $(x,y)-plane$ through transformation

$$x = x(\varphi,\psi)\,, \qquad y = y(\varphi,\psi) \tag{14}$$

such that the Jacobian $J = \dfrac{\partial(x,y)}{\partial(\varphi,\psi)}$ of the transformation is non-zero and finite. Follow Martin [15], let $\alpha$ be the angle between the tangent to the streamlines $\psi = const.$ and the curves $\varphi = const.$ at a point $P(x,y)$, then

$$\tan(\alpha) = \frac{y_\varphi}{x_\varphi} \tag{15}$$

$$ds^2 = E(\varphi,\psi)\, d\varphi^2 + 2\, F(\varphi,\psi)\, d\varphi\, d\psi + G(\varphi,\psi)\, d\psi^2 \tag{16}$$

and

$$E = x_\varphi^2 + y_\varphi^2\,, F = x_\varphi\, x_\psi + y_\varphi\, y_\psi\,, G = (x_\psi)^2 + (y_\psi)^2 \tag{17}$$

Differentiating equation (14) with respect to $x$ and $y$, and solving the resulting equations

$$y_{\varphi}=-J\,\psi_{x},\quad y_{\psi}=J\,\varphi_{x},\quad x_{\varphi}=J\,\psi_{y},\quad x_{\psi}=-J\,\varphi_{y} \tag{18}$$

and

$$J=\pm\sqrt{EG-F^{2}}=\pm(x_{\varphi}\,y_{\psi}-y_{\varphi}\,x_{\psi})=\pm W \tag{19}$$

Trigonometric identities on equation (15) and equation (18) provides

$$x_{\varphi}=\sqrt{E}\,Cos(\alpha),\quad x_{\psi}=\frac{1}{\sqrt{E}}[F\,Cos(\alpha)-J\,Sin(\alpha)]$$

$$y_{\varphi}=\sqrt{E}\,Sin(\alpha),\quad y_{\psi}=\frac{1}{\sqrt{E}}[F\,Sin(\alpha)+J\,Cos(\alpha)] \tag{20}$$

The integrability conditions:

$$x_{\psi\varphi}=x_{\varphi\psi}\qquad\qquad y_{\psi\varphi}=y_{\varphi\psi} \tag{21}$$

for $x$ and $y$, yield:

$$\alpha_{\varphi}=\frac{J\,\Gamma_{11}^{2}}{E},\quad \alpha_{\psi}=\frac{J\,\Gamma_{12}^{2}}{E} \tag{22}$$

wherein

$$\Gamma_{11}^{2}=\frac{1}{2W^{2}}[-FE_{\varphi}+2EF_{\varphi}-EE_{\psi}]\quad,\ \Gamma_{12}^{2}=\frac{1}{2W^{2}}[EG_{\varphi}-FE_{\psi}] \tag{23}$$

Equation (21), applying the integrability condition $\alpha_{\varphi\psi}=\alpha_{\psi\varphi}$ for $\alpha(\varphi,\psi)$, yields

$$K=\frac{1}{W}\left[\left(\frac{W\,\Gamma_{11}^{2}}{E}\right)_{\psi}-\left(\frac{W\,\Gamma_{12}^{2}}{E}\right)_{\varphi}\right] \tag{24}$$

where K is called the Gaussian curvature.

Now equations (10-11), on substituting equation (15), equation (18), equation (20) , equations (22-23) simplifies to following

$$-R_{e}\Omega JE=R_{e}\,JEL_{\psi}+A_{\varphi}\left((F^{2}-J^{2})\cos 2\alpha-2FJ\sin 2\alpha\right)$$
$$+EA_{\psi}\left(J\sin 2\alpha-F\cos 2\alpha)\right)-B_{\varphi}\left(\frac{1}{2}(F^{2}-J^{2})\sin 2\alpha+FJ\cos 2\alpha\right)$$

$$+E\,B_{\psi}\left(\frac{1}{2}F\sin 2\alpha + J\cos^2\alpha\right) \tag{25}$$

and

$$0 = -R_e J\,L_{\varphi} + E\,A_{\psi}\cos 2\alpha - A_{\varphi}\left[F\cos 2\alpha - J\sin 2\alpha\right]$$

$$+\ B_{\varphi}\left(\frac{1}{2}F\sin 2\alpha - J\sin^2\alpha\right) - \frac{E\,B_{\psi}}{2}\sin 2\alpha \tag{26}$$

Differential geometry [23], says that

$$(T_{xx} + T_{yy}) = \frac{1}{J}\left[\left(\frac{G\,T_{\varphi} - F\,T_{\psi}}{J}\right)_{\varphi} + \left(\frac{E\,T_{\psi} - F\,T_{\varphi}}{J}\right)_{\psi}\right] \tag{27}$$

The expression $u\,T_x + v\,T_y$ becomes $\frac{T_{\varphi}}{J}$, thus equation (12) on using (27) becomes

$$\frac{1}{J\,P_{e'}}\left[\left(\frac{G\,T_{\varphi} - F\,T_{\psi}}{J}\right)_{\varphi} + \left(\frac{E\,T_{\psi} - F\,T_{\varphi}}{J}\right)_{\psi}\right] = -\frac{\mathrm{E_c}}{\mathrm{R_e}}\,\frac{1}{4\mu}\left(B^2 + 4A^2\right) + \frac{T_{\varphi}}{J} \tag{28}$$

and

$$q = \frac{\sqrt{E}}{J}$$

(29)

where $q = \sqrt{u^2 + v^2}$ .

Equation (13) on substitute equations (18-23), provides

$$B(\varphi,\psi) = \frac{4\mu}{EJ^3}\,[\,E_{\varphi}\,(F\sin\alpha + J\cos\alpha)^2 - 2\,E(F\sin\alpha + J\cos\alpha)$$

$$(F_{\varphi}\sin\alpha + J_{\varphi}\cos\alpha)\ +\ E^2(J_{\psi}\sin 2\alpha + G_{\varphi}\sin^2\alpha)\,]\ , \tag{30}$$

$$A(\varphi,\psi) = \mu\,[-\ \frac{(F\cos\alpha - J\sin\alpha)}{4E^2\,J^5}$$

$$\{E_{\varphi}(2E\,J^3\cos\alpha + F\sqrt{E}\sin\alpha)$$

$$-\ 4E^2J^2J_{\varphi}\cos\alpha\ -\ 2E\sqrt{E}\,F_{\varphi}\sin\alpha\ +\ E\sqrt{E}\,E_{\psi}\sin\alpha\}$$

$$+\frac{\cos\alpha}{2J^3}\{E_\psi(F\sin\alpha+J\cos\alpha)- \quad 2EJ_\psi\cos\alpha-EG_\varphi\sin\alpha\}$$

$$+\frac{(F\sin\alpha+J\cos\alpha)}{2EJ^3}\{(JE_\varphi-2EJ_\varphi)\sin\alpha$$

$$+\cos\alpha\ [-FE_\varphi+2EF_\varphi-EE_\psi]\}$$

$$-\frac{\sin\alpha}{2J^3}\{(E_\psi(J\sin\alpha-F\cos\alpha)-2EJ_\psi\sin\alpha+EG_\varphi\cos\alpha\}] \quad (31)$$

In Martin's system, equation (8) becomes

$$\Omega=v_\varphi\varphi_x+v_\psi\psi_x-u_\varphi\varphi_y-u_\psi\psi_y \quad (32)$$

Equation (32) on substituting equation (15), equation (18) and equation (20) provides

$$\Omega=\frac{(F\sin\alpha+J\cos\alpha)}{2EJ^3}\{(JE_\varphi-2EJ_\varphi)\sin\alpha+ \quad \cos\alpha$$

$$[-FE_\varphi+2EF_\varphi-EE_\psi]\}$$

$$-\frac{\sin\alpha}{2J^3}\{E_\psi(J\sin\alpha-F\cos\alpha)-2EJ_\psi\sin\alpha+EG_\varphi\cos\alpha\}]$$

$$+\frac{(F\cos\alpha-J\sin\alpha)}{4E^2J^5}\{E_\varphi(2EJ^3\cos\alpha+F\sqrt{E}\sin\alpha)$$

$$-4E^2J^2J_\varphi\cos\alpha-2E\sqrt{E}F_\varphi\sin\alpha+E\sqrt{E}E_\psi\sin\alpha\}$$

$$-\quad [\frac{\cos\alpha}{2J^3}\{E_\psi(F\sin\alpha+J\cos\alpha)- \quad 2EJ_\psi\cos\alpha-$$

$$EG_\varphi\sin\alpha\}] \quad (33)$$

The basic system of non-dimensional partial differential equations for the problem concerned in Martin's system comprises of equations (25-26), equations (28), with equations (30-31) and equation (33).

## 3. Basic equations in von-Mises coordinates

In order to achieve the objective of this discourse set

$$\varphi=x \quad (34)$$

The equation (15), equation (17), equations (19), equations (30-31) and equation (33) reduces to

$$\cos\alpha = \frac{1}{\sqrt{E}} \tag{35}$$

$$E = 1 + \left(x\, f'(x)\right)^2 \tag{36}$$

$$J = a\, x \tag{37}$$

$$B = \frac{-4\mu}{a\,x^2} \tag{38}$$

$$A = \frac{\mu}{a\,x^2}\left(x\left(x\, f'(x)\right)' - 2x\, f'(x)\right) \tag{39}$$

$$\Omega = \frac{\left(x\, f'(x)\right)'}{a\,x} \tag{40}$$

The equations (25-26) and equation (28) on utilizing equations (34-40), give

$$-R_e\,\Omega = R_e\, L_\psi - a\,x\, A_x + M\, A_\psi + B_\psi \tag{41}$$

$$0 = -R_e\, L_x + \frac{A_\psi (1 - M^2)}{a\,x} + M\, A_x - \frac{M\, B_\psi}{a\,x} \tag{42}$$

$$a\,x T_{xx} - 2M\, T_{\psi x} + \frac{(1+M^2)}{a\,x} T_{\psi\psi} + \left(a - P_{e'}\right) T_x + M' T_\psi$$

$$= -\frac{a\,x E_c\, P_r}{4\mu}\left(B^2 + 4A^2\right) \tag{43}$$

and

$$M = x\, f'(x) \tag{44}$$

Considering $L_{x\psi} = L_{\psi x}$ on equations (41-42) yields

$$a\,x A_{xx} - 2M\, A_{x\psi} - \frac{\left[1 - M^2\right]}{a\,x} A_{\psi\psi} + a\, A_x - A_\psi\, M' - \left\{B_x - \frac{f' B_\psi}{a}\right\}_\psi = R_e\, w_x \tag{45}$$

Finding solution of the equation (45), the generalized energy function $L$ from equation (41), temperature distribution $T$ equations (42), $\mu$ from equation (38) or equation (39), the velocity vector

$\mathbf{q}=(u,v)$ from equation (7), $p$ from equation (9), and streamlines from equation (2) are determined.

**4. Exact solutions**The compatibility equation (45) involves the functions $A$, $B$ and derivative of $f(x)$. For its solution this discourse eliminates $\mu$ from the function $A$ and $B$. The equation (38) and equation (39), on eliminate $\mu$, provides

$$A= X(x)\ B \tag{46}$$

where

$$X(x)=\frac{-1}{4}\left(xM'-2M\right) \tag{47}$$

provided $\left(xM'-2M\right)\neq 0$.

Use of equation (46) in equation (45), gives

$$a\,x\,X\,B_{xx} \quad -(1+2M\,X) \quad B_{x\psi}+ \quad \frac{B_{\psi\psi}}{a\,x}\left\{M-X(1-M^2)\right\} \quad +$$

$$a\,B_x\left\{2\,x\,X'+X\right\}$$

$$-\ B_\psi\left(2MX'+M'X\right)+\ a\,B\left(x\,X'\right)' \quad =R_e\left(\frac{M'}{a\,x}\right)' \tag{48}$$

Since the coefficients of $B_{xx}$, $B_{x\psi}$, $B_{\psi\psi}$, $B_x$, $B_\psi$ and $B$ are all functions of $x$ only therefore the solution of equation (48) is of the form

$$B= G(x)+\ K(x)\ \psi \tag{49}$$

where $G(x)$ and $K(x)$ are to be determined.

Equation (48) on substituting equation (49) gives

$$\psi\ [x\ X\ K''(x)+(X+2\ x\ X')K'(x)\ +(X'+x\ X'')K(x)]$$

$$+[a\,x\,X\ \ G''(x)\ +\ a\left(X+2\ r\,X'\right)G'(x)+\ a\left(x\,X'\right)'\ G(x)]$$

$$= R_e \left( \frac{M'}{a\,x} \right)' + a\,(1 + 2M\,X)\;K'(x) + a\,\left(2MX' + M'X\right)\;K(x) \quad (50)$$

Since $x$ and $\psi$ are von-Mises coordinates therefore equation (50) provides

$$x\;X\;K''(x) + (X + 2\;x\;X')\,K'(x) + (X' + x\;X'')\,K(x) = 0 \quad (51)$$

and

$$a\,x\,X\;\;G''(x) + a\left(X + 2\,r\,X'\right)G'(x) + a\left(x\,X'\right)'\,G(x) = Z_1(r) \quad (52)$$

where

$$Z_1(x) = R_e \left( \frac{M'}{a\,x} \right)' + a\,(1 + 2M\,X)\;K'(x) + a\,(2MX' + M'X)\;K(x) \quad (53)$$

Equation (51) in exact differential form is

$$\frac{d}{dx}\left[ x \frac{d\,(X\,K)}{dx} \right] = 0 \quad (54)$$

Equation (54) implies

$$K(x) = \frac{k_1 \ln x}{X(x)} + \frac{k_2}{X(x)} \quad (55)$$

where $k_1$ and $k_2$ are constants.

Equation (52) in exact differential form is

$$\frac{d}{dx}\left[ x \frac{d\,(X\,G)}{dx} \right] = Z_1(x) \quad (56)$$

Equation (56) implies

$$G(x) = \frac{1}{X} \int \;[\frac{1}{x} \int Z_1(x)\,dx]\;\,dx + \frac{k_3 \ln x + k_4}{X} \quad (57)$$

where $k_3$ and $k_4$ are constants.

On substituting equation (55) and equation (57) in equation (49), one can have

$$B = \frac{1}{X} \int \;[\frac{1}{x} \int Z_1(x)\,dx]\;\,dx + \frac{k_3 \ln x + k_4}{X} + \left( \frac{k_1 \ln x + k_2}{X} \right) \psi \quad (58)$$

Viscosity from either from equation (40) or equation (41) is

$$\mu(x,\psi) = \frac{-a\,x^2}{4}\,[\,\frac{1}{X} \int \;[\frac{1}{x} \int Z_1(x)\,dx]\;\,dx + \frac{k_3 \ln x + k_4}{X} + \left( \frac{k_1 \ln x + k_2}{X} \right) \psi\,] \quad (59)$$

The solution of equations (41-42) implies the following expression of function $L$

$$a R_e L = -R_e \left(\frac{M'}{a x}\right)\psi + a x (G X)' \psi - a K(M X + 1)\psi + a x (K X)' \left(\frac{\psi^2}{2}\right)$$

$$+ a\int M (G X)' dx + \int\left[\frac{K}{x}\left\{a X (1 - M^2) - a M\right\}\right] dx + p_1 \qquad (60)$$

The temperature distribution $T$ is determined from equation (38). Equation (43) on utilizing equations (58-59), become

$$a x T_{xx} - 2 M T_{\psi x} + \frac{(1+M^2)}{a x} T_{\psi\psi} + (a - P_{e'}) T_x + M' T_\psi$$

$$= \frac{\mathrm{E_c\,P_r}(1 + 4X^2)}{x}\{G(x) + K(x)\,\psi\} \qquad (61)$$

Equation (61) suggests seeking solution of the form

$$T = T_1(x) + T_2(x)\,\psi \qquad (62)$$

The use of equation (62) in equation (61), gives

$$T_\alpha'' + \frac{(a - P_{e'})}{a x} T_\alpha' = Z_{\alpha+1}(x), \qquad \text{for} \qquad \alpha \in \{1, 2\}, \qquad (63)$$

where

$$Z_2(x) = \frac{\mathrm{E_c\,P_r}(1 + 4X^2)}{a x^2} G(x) + 2\,M\,T_2' + M'\,T_2 \qquad (64)$$

$$Z_3(x) = \frac{\mathrm{E_c\,P_r}(1 + 4X^2)}{a x^2} K(x). \qquad (65)$$

The solution of equations (63) and (64) are

$$T_\alpha(x) = \int x^{\frac{-(a-P_{e'})}{a}}\left[\int\left\{x^{\frac{(a-P_{e'})}{a}} Z_{\alpha+1}(x)\right\}dx\right]dx + H_\alpha \int x^{\frac{-(a-P_{e'})}{a}} dx + H_{\alpha+2} \quad (67)$$

where $H_i$, $i = 1,2,\ldots 4$ are constant.

Therefore, temperature distribution moderate Peclet number is obtained by substituting equations (64-67) in equation (62) for $\alpha \in \{1, 2\}$. Thus, exact solutions for arbitrary $f(x)$ is found.

Now $(x M' - 2M) = 0$ implies $A = 0$ and

$$f(x) = \frac{1}{2} c_1 x^2 + c_2 \tag{68}$$

The compatibility equation (48) on substituting (68) provides

$$\left\{ B_x - \frac{f' B_\psi}{a} \right\}_\psi = 0 \tag{69}$$

The solution of equation (69) is

$$B = \frac{c_1 b_1}{2a} x^2 + b_1 \psi + \int I(x)\, dx + c_3 \tag{70}$$

Where $c_1, c_2$, $c_3$, $b_1$ are constants and $I(x)$ is function of intergation.

Equation (41) utilization of equation (70) provides

$$\mu = \frac{-a x^2}{4} \left[ \frac{c_1 b_1}{2a} x^2 + b_1 \psi + \int I(x)\, dx + c_3 \right] \tag{71}$$

The solution of equations (41-42) is

$$\text{Re } L = \left( -b_1 - \frac{2 c_1 \text{Re}}{a^2} \right) \psi - \frac{c_1 b_1}{2a} x^2 + M_0 \tag{72}$$

where $M_0$ is a real constant.

The energy equation (43) on employing equations (69), equations (71-72), becomes

$$a\, x^2 T_{xx} - 2 c_1 x^3 T_{\psi r} + \frac{x}{a}(1 + c_1^2 x^4) T_{\psi\psi} - 2 c_1 x^2 T_\psi + x\,(a - P_{e'})\, T_r$$

$$= \text{E}_\text{c} \text{P}_\text{r} \left[ \frac{c_1 b_1}{2a} x^2 + b_1 \psi + \int I(x)\, dx + c_3 \right] \tag{73}$$

Equation (73) suggests

$$T = A_1 \psi^3 + T_5(x) \psi^2 + T_4(x) \psi + T_3(x) \tag{74}$$

Equation (73), employing equation (74), provides

$$a\ x^2\, T_5'' + x\,(a - P_{e'})\, T_5' = 6\,a\,c_1\,A_1\,x^2 \tag{75}$$

$$a\ x^2\, T_4'' + x\,(a - P_{e'})\, T_4' = 4\,a\,c_1\,x^3\,T_5' + 4\,a\,c_1\,x^2\,T_5$$
$$-\ 6\,a\,A_1\,x(1 + c_1^2 x^4) + b_1 E_c\, P_r \tag{76}$$

$$a\ x^2\, T_3'' + x\,(a - P_{e'})\, T_3' = 2\,a\,c_1\,x^3\,T_4' + 2\,a\,c_1\,x^2\,T_4 -$$
$$2\,a\ x(1 + c_1^2 x^4)\,T_5 + E_c\, P_r \left( c_{14} + \frac{c_1 b_1}{2\,a}\,x^2 + \int I(x)\,dx \right) \tag{77}$$

When $(a - P_{e'}) \neq 0$ the equations (75-77) give

$$T_5(x) = \frac{3a\,A_1 c_1\,x^2}{(2a - P_{e'})} + \frac{a\,n_1}{P_{e'}}\,x^{\frac{P_{e'}}{a}} + n_2\,, \quad (2a - R_e\,P_r) \neq 0 \tag{78}$$

$$T_4(x) = -\frac{6a\,A_1 c_1^2\,x^5}{5(5a - P_{e'})} - \frac{6a\,A_1\,x}{(a - P_{e'})} + \frac{9a^2\,A_1\,c_1^2\,x^4}{(4a - P_{e'})(2a - P_{e'})}$$
$$+ \frac{2a\,c_1\,x^2\,n_2}{(2a - P_{e'})} + \frac{a\,n_1}{P_{e'}}\,x^{\frac{P_{e'}}{a}} + n_4 - \frac{b_1 E_c\ \ln x}{R_e}$$
$$+ \frac{a\,n_1\,x^{\frac{P_{e'}}{a}}\left[2a\,(c_1\,x^2 n_1 + n_3) + P_{e'}(2\,c_1\,x^2 n_1 + n_3)\right]}{P_{e'}(2a - P_{e'})},$$

where $(2a - P_{e'}) \neq 0\,,\ (4a - P_{e'}) \neq 0\,, (5a - P_{e'}) \neq 0$ (79)

$$T_3(x) = n_6 + \int M_1(x)\,dx \tag{80}$$

$$M_1(x) = n_5\,x^{-\left\{\frac{(a - P_{e'})}{a}\right\}}$$
$$+\ x^{-\left\{\frac{(a - P_{e'})}{a}\right\}} \int \left\{ E_c\, P_r \left( c_{14} + \frac{c_1 b_1}{2\,a}\,x^2 + \int I(x)\,dx \right) \right\} x^{\left\{\frac{(a - P_{e'})}{a}\right\}} dx$$
$$+\ x^{-\left\{\frac{(a - P_{e'})}{a}\right\}} \int \left\{ 2\,a\,c_1\,x^2\,T_4' + 2\,a\,c_1\,x^2\,T_4 - 2\,a\,x(1 + c_1^2\,x^4)\,T_5 \right\} x^{\left\{\frac{(a - P_{e'})}{a}\right\}} dx$$
(81)

and $n_1,\ n_2,\ n_3,\ n_4, n_5, n_6$ are non zero constants.

Therefore, temperature distribution is obtained by substituting equations (78-81) in equation (74) for moderate Peclet number. Thus, exact solutions for $f(x)$ given by equation (68) is found when $(a-P_{e'})\neq 0$.

When $(a-P_{e'})=0$, equations (76–78) yield

$$T_5(x)=3c_1 A_1 x^2+n_7 x+n_8 \tag{82}$$

$$T_4(x)=-\frac{3A_1 c_1^2 x^5}{10}+3c_1^2 A_1 x^4+\frac{4}{3}c_1 n_7 x^3+2c_1 n_8 x^2$$
$$-\ 6A_1 x\ln x+(6A_1+n_9)x-\frac{b_1 E_c P_r}{a}\ln x+n_{10} \tag{83}$$

$$T_3(x)=\iint M_2(x)\,dx\,dx+n_{11}x+n_{12} \tag{84}$$

where

$$M_2(x)=2c_1 xT_4'+2c_1 x-\frac{2}{x}(1+c_1^2 x^4)T_5(x)$$
$$+\frac{E_c P_r}{a x^2}\left[c_{14}+\frac{c_1 b_1}{2}x^2+\int I(x)\,dx\right] \tag{85}$$

and $n_7, n_8\ n_9, n_{10}\ n_{11}, n_{12}$ are non zero constants.

Therefore, temperature distribution is obtained by substituting equations (82-85) in equation (74) for moderate Peclet number. Thus, exact solutions, for $f(x)$ given by equation (68), is found when $(a-P_{e'})=0$.

## 5. Conclusion

The following dimensionless parameters

$$x^*=\frac{x}{L_0}\qquad y^*=\frac{y}{L_0}\qquad u^*=\frac{u}{U_0}\qquad v^*=\frac{v}{U_0}$$

$$\mu^* = \frac{\mu}{\mu_0} \qquad p^* = \frac{p}{p_0} \qquad F_1^* = \frac{F_1}{F_0} \qquad F_2^* = \frac{F_2}{F_0}$$

where the thermal conductivity $k = k_0 = Const.$, density $\rho = \rho_0 = Const.$ and $c_v = c_p = Const.$ is used for writing the basic equations governing the two-dimensional steady motion of incompressible fluid of variable viscosity in non-dimensional form.

Successive transformation technique is used in finding a class of new exact solutions of the basic non-dimensional equations with moderate Peclet number in von-Mises coordinates. The exact solutions are determined for arbitrary and non-arbitrary $f(x)$. For arbitrary $f(x)$ the streamlines $\psi = const.$ are $y - f(x) - b = \psi$ and for $f(x) = \frac{1}{2}c_1 x^2 + c_2$ the streamlines are $\left[y - \frac{1}{2}c_1 x^2 + c_2 - b\right] = \psi$ where $c_1$ and $c_2$ are constants. In both the cases an infinite set of velocity components, viscosity function, generalized energy function and temperature distribution for moderate Peclet number can be constructed. The graph of streamlines can be drawn using computer algebra system software to observe the streamline patterns.

## References

[1] Chandna, O. P., Oku-Ukpong E. O.; Flows for chosen vorticity functions-Exact solutions of the Navier-Stokes Equations: *International Journal of Applied Mathematics and Mathematical Sciences*, **17(1)** (1994) 155-164.

[2] Naeem, R. K.; Exact solutions of flow equations of an incompressible fluid of variable viscosity via one – parameter group: *The Arabian Journal for Science and Engineering*, 1994, **19**(1), 111-114.

[3] Naeem, R. K.; Srfaraz, A. N.; Study of steady plane flows of an incompressible fluid of variable viscosity using Martin’s System: *Journal of Applied Mechanics and Engineering*, 1996, **1**(1), 397-433.

[4] Naeem, R. K.; Steady plane flows of an incompressible fluid of variable viscosity via Hodograph transformation method: *Karachi University Journal of Sciences*, 2003, **3(1)**, 73-89.

[5] Naeem, R. K.; On plane flows of an incompressible fluid of variable viscosity: *Quarterly Science Vision*, 2007, **12(1)**, 125-131.

[6] Landau L. D. and Lifshitz E. M.; Fluid Mechanics, *Pergmaon Press*, vol 6

[7] Naeem, R. K.; Mushtaq A.; A class of exact solutions to the fundamental equations for plane steady incompressible and variable viscosity fluid in the absence of body force: *International Journal of Basic and Applied Sciences*, 2015, **4(4),** 429-465. www.sciencepubco.com/index.php/IJBAS ,doi:10.14419/ijbas.v4i4.5064

[8] Giga, Y.; Inui, K.; Mahalov; Matasui S.; Uniform local solvability for the Navier-Stokes equations with the Coriolis force: *Method and application of Analysis*, 2005, 12 , 381-384.

[9] Gerbeau, J. -F.; Le Bris, C., A basic Remark on Some Navier-Stokes Equations With Body Forces: *Applied Mathematics Letters*, 2000, **13(1)**, 107-112.

[10] Mushtaq A., On Some Thermally Conducting Fluids: *Ph. D Thesis, Department of Mathematics, University of Karachi, Pakistan*, 2016.

[11] Mushtaq A.; Naeem R.K.; S. Anwer Ali; A class of new exact solutions of Navier-Stokes equations with body force for viscous incompressible fluid,: *International Journal of Applied Mathematical Research*, 2018, **7(1),** 22-26. www.sciencepubco.com/index.php/IJAMR,doi:10.14419/ijamr.v7i1.8836

[12] Mushtaq Ahmed, Waseem Ahmed Khan : A Class of New Exact Solutions of the System of PDE for the plane motion of viscous incompressible fluids in the presence of body force,: *International Journal of Applied Mathematical Research*, 2018, **7 (2)** , 42-48. www.sciencepubco.com/index.php/IJAMR, doi:10.14419 /ijamr.v7i2.9694

[13] Mushtaq Ahmed, Waseem Ahmed Khan , S. M. Shad Ahsen : A Class of Exact Solutions of Equations for Plane Steady Motion of Incompressible Fluids of Variable viscosity in presence of Body Force,: *International Journal of Applied Mathematical Research*, 2018, **7 (3)** , 77-81.

www.sciencepubco.com/index.php/IJAMR,
doi:10.14419/ijamr.v7i2.12326
[14] Mushtaq Ahmed, (2018), A Class of New Exact Solution of equations for Motion of Variable Viscosity Fluid In presence of Body Force with Moderate Peclet number , *International Journal of Fluid Mexhanics and Thermal Sciences,* 4 (4) 429- www.sciencepublishingdroup.com/j/ijfmts
doi: 10.11648/j.ijfmts.20180401.12
[15] Martin, M. H.; The flow of a viscous fluid I: *Archive for Rational Mechanics and Analysis*, 1971, **41**(4), 266-286.
[16] Daniel Zwillinger; Handbook of differential equations; *Academic Press, Inc.* (1989)
[17] R. Kronig and J.C. Brink, On the Theory of Extraction from Falling Droplets, *Appl. Sci. Res. Ser. A*, 2:142-154, 1950.
[18] B. Abramzon and C. Elata, Numerical analysis of unsteady conjugate heat transfer between a single spherical particle and surrounding flow at intermediate Reynolds and Peclet numbers, *2nd Int. Conf. on numerical methods in Thermal problems, Venice*, pp. 1145-1153,1981.
[19] D.L.R. Oliver & K.J. De Witt, High Peclet number heat transfer from a droplet suspended in an electric field: Interior problem, *Int. J. Heat Mass Transfer*, vol. 36: 3153-3155, 1993.
[20] Z.G. Feng, E.E. Michaelides, Unsteady heat transfer from a spherical particle at finite Peclet numbers, *J. Fluids Eng.* 118: 96-102, 1996.
[21] Z.G. Fenz, E.E. Michaelides, Unsteady mass transport from a sphere immersed in a porous medium at finite Peclet numbers, *Int. J. Heat Mass Transfer* 42: 3529-3531, 1999.
[22] Fayerweather Carl, Heat Transfer From a Droplet at Moderate Peclet Numbers with heat Generation. *PhD. Thesis, U of Toledo*, May 2007.
[23] Weatherburn C.E., Differential geometry of three dimensions, *Cambridge University Press*, 1964