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The Applications of Tangent Bundles in Human-Robot Interactions

ABDEL RADI ABDEL RAHMAN ABDEL GADIR
Department of Mathematics, Faculty of Education
Omdurman Islamic University, Omdurman, Sudan
RAGAA MOHAMMED HAJ IBRAHIM
Department of Mathematics, Faculty of Education
Elzaiem ElAzhary University, Omdurman, Sudan
NEDAL HASSAN ELBADOWI ELJANEID
Department of Mathematics, College of Science
Tabuk University, Saudi Arabia

Abstract

This research aims at applying human-robot interactions in tangent bundles. We used the historical analysis mathematical method, then we found the following results: The concept of the tangent space as a basic concept of the tangent bundles, we came to the concept of fiber bundles from a general perspective, that tangent bundle have a great role in many engineering fields and we applied the human-robot interactions in tangent bundles to clarify their importance in modern science.

Key Words: Fiber bundles, Tangent Bundles, Topological Space

1.INTRODUCTION:

The tangent space $T_x M^n$ can be defined for any point x of a smooth manifold M^n. Our next problem is to construct a topological space and even a smooth manifold from all vectors of this family of vector spaces that depend on the pointx.

Considering the disjoint union $TM^n = \bigcup T_x M^n$ of all tangent spaces to a manifold M^n , we define the projection $\pi : TM^n \to M^n$ by mapping each vector from $T_x M^n$ into the point x .then $\pi^{-1}(x) = T_x M^n$

This inverse image is called the fiber over the point x. [15]

For any smooth manifold M , we define the tangent bundle of M , denoted by TM , to be the disjoint union of the tangent spaces at all points of M

$$TM = \cup_{p\in M} TM \quad [7]$$

2. Tangent Space:

We can consider the tangent space at a point $p \in M$ of a manifold: let us assume that the manifold is embedded in Euclidean space $\mathbb{R}^n$, and then it is quite obvious that to every point $p \in M$ there is assigned a certain linear subspace of $\mathbb{R}^n$.

The space of tangent vectors of M at x, the velocity vectors of possible movements on M . Thus the sphere S^n is embedded in $\mathbb{R}^{n+1}$ as $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$, and the tangent space at the point $x \in S^n$ is the set of vectors $\{v \in \mathbb{R}^{n+1} \mid < v, x >= 0\}$ .In general such and embedding is not canonically given, we must describe the tangent space by the intrinsic properties of the manifold. [12].

Lemma(2.1) : let M be a C^k -manifold , let $\mathrm{m} \in \mathrm{M}$ and let y_1, y_2 be curves of class C^k through m . If $\frac{d}{dt} \restriction_0 (\mathrm{k} \circ y_1)(t) = \frac{d}{dt} \restriction_0 (\mathrm{k} \circ y_2)(t)$ for some local chart (U, k) at m, then this holds for every such chart .

Proof : we can define two curves y_1, y_2 in $k_m(M)$ to be equivalent if for some (and hence any) local chart (U, k) at m there holds

$$\frac{d}{dt} \restriction_0 (\mathrm{k} \circ y_1)(t) = \frac{d}{dt} \restriction_0 (\mathrm{k} \circ y_2)(t) \qquad (2.1)$$

Let T_mM denote the set of equivalence classes on T_mM , a linear structure can be defined as follows . Let (U, k) be a local chart at m . The mapping :

$$F_m^k : T_mM \to R^n , [y] \to F_m^k[y] := \frac{d}{dt}\restriction_0 (k \circ y)(t) \quad (2.2)$$

Is well defined and injective . Since $x \in R^n$ is the image under F_m^k of the equivalence class of the curve

$$y^x(t) := k^{-1}(k(m) + tx), \quad (2.3)$$

F_m^k is also subjective . The inverse mapping is given by $(F_m^k)^{-1}(x) = [y^x]$

By means of F_m^k , we transport the linear structure of R^n to T_mM , that is we define

$$\alpha[y_1] + \beta[y_2] = (F_m^k)^{-1}\left(\alpha F_m^k([y_1]) + \beta F_m^k([y_2])\right) \quad (2.4)$$

This definition does not depend on the choice of chart , because for a second chart (V, p) one has

$$F_m^p([y]) = (p \circ k^{-1})'\left(k(m)\right). F_m^k([y]). \quad (2.5)$$

T_mM is a real linear space of the same dimension as M and the mappings F_m^k are vector space isomorphism . [6]

Example(2.2): Let M be an open subset of a finite-dimensional real vector space V. For every $v \in M$, the assignment of the velocity vector $\frac{d}{dt}\restriction_0 y(t) \in V$ to the class $[y] \in T_vM$ defines an isomorphism of the tangent space T_vM with the ambient vector space V. The inverse of this isomorphism is obtained by assigning to $u \in V$ the class of the curve $t \to v + tu$. This isomorphism will be referred to as the natural identification of T_vM with V . [6]

(2.3)Partial Derivatives:

Definition (2.2.1): Let $F : N \to M$ be a smooth map, and let $(U,\varphi) = (U, x^1, \ldots, x^n)$ and $(V,\psi) = (V, y^1, \ldots, y^m)$ be charts on N and M respectively such that $(U) \subset V$. Denote by

$$F^i := y^i \circ F = r^i \circ \psi \circ F : U \to R,$$

the ith component of F in the chart (V,ψ). Then the matrix $[\partial F^i/\partial x^j]$ is called the Jacobian matrix of F relative to the charts (U,φ) and (V,ψ). In case N and M have the same dimension, the determinant $det[\partial F^i/\partial x^j]$ is called the Jacobin determinant of F relative to the two charts. The Jacobin determinant is also written as $\partial(F_1,\ldots,F_n)/\partial(x_1,\ldots,x_n)$.[9]

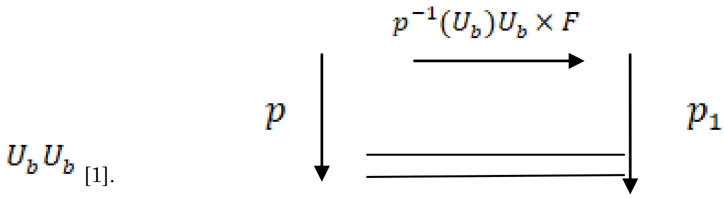
3. Fiber Bundles:

Recall that tangent and cotangent bundles, TM and T^*M , are special cases of a more general geometrical object called fiber bundle , where the word fiber V of map $\pi: Y \to X$ denotes the pre image $\pi^{-1}(x)$ of an element $x \in X$.

We can say that a fiber bundle Y is a homomorphism generalization of a product space $X \times V$ where X and V are called the base and the fiber respectively $\pi: Y \to X$ is called the projection , $Y_x = \pi^{-1}(x)$ denote a fiber over a point x of the base X , while the map $f = \pi^{-1}: X \to Y$ defines the cross-section , producing the graph $(x, f(x))$ in the bundle $Y. (f = \dot{x})$. [13]

Definition (3.1): A manifold X is called a fiber bundle over a base B with a fiber F if a smooth surjective map $p: X \to B$ is given such that locally X is a direct product of a part of the base and the fiber. More precisely, we require that any point $b \in B$ has a neighborhood U_p such that $p^{-1}(U_p)$ can be identified with $U_b \times F$ via a smooth map α_b

so that the following diagram (where p_1 denotes the projection to the first factor) is commutative:

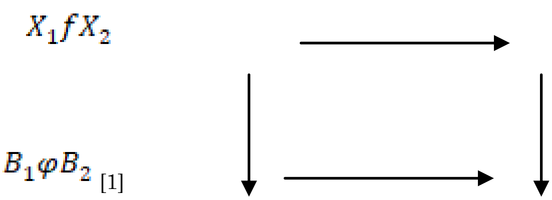


[1].

"Fig 1"

Notation (3.2): Sometimes fiber bundles are called twisted or skew products of B and F. The reason is that a direct product $X = B \times F$ is a particular kind of fiber bundle: one can put $U = B, a = Id$. We observe also that B and F play non-symmetric roles in the construction of a skew product.

The collection of all fiber bundles forms a category where the morphemes from $X_1 \xrightarrow{F_1} B_1$ to $X_2 \xrightarrow{F_2} B_2$ are pairs of smooth neaps $(f : X_1 \rightarrow X_1 , \varphi: B_1 \rightarrow B_2)$ such that the following diagram is commutative:



[1]

" Fig 2"

Definition(3.3)(Section): A map $S: B \rightarrow X$ is a section of a fiber bundle $X \xrightarrow{F} B$ if $p \circ S = Id$. In the case of a trivial bundle the sections can be identified with functions $f: B \rightarrow F$. So, the section of a fiber bundle $X \xrightarrow{F} B$ gives a natural generalization of the notion a function from $B to F$. [1]

4. Vector Bundles

Definition (4.1): A fiber bundle (X, B, F, p) is called a vector bundle if F and all F_b , $b \in B$, are vector spaces (real or complex) and all maps α_b , $b \in B$, are linear on the fibers .[1]

Definition (4.2): A bundle atlas for a vector over a differential manifold is differentiable if all its transition functions are differentiable. A differentiable vector bundle is a pair (E, B) consisting of a vector bundle E over M and maximal differentiable bundle atlas B for E .

We often encounter differentiable and topological vector bundles in a form in which one could perhaps call them $"prevectorbundle"$ given are the usual defining terms.

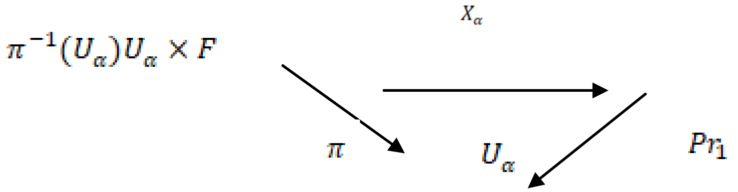
E : Total space, π : Projection, X : Base , B : Bundle atlas.[12]

Definition (4.3): Let $K = R or C$ and $K \geq 0$. A K-vector bundle of class C^k is a triple (E, M, π),where E and M are C^k-manifolds and $\pi: E \to M$ is a surjective C^k-mapping satisfying the following conditions :

1.Forevery $m \in M$, $E_m := \pi^{-1}(m)$ carries the structure of a vector space over K.

2.There exists a finite-dimensional vector space F over K, an open covering $\{U_\alpha\}$ of Man associated family of C^k-diffeomorphism $X_\alpha: \pi^{-1}(U_\alpha) \to U_\alpha \times F$ such that, for all α,

(a) The following diagram commutes,



" Fig 3"

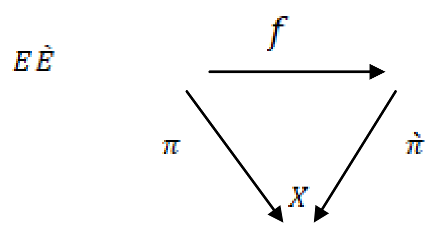
(b) For every $m \in U_\alpha$, the induced mapping
$X_{\alpha,m} := Pr_2 \circ X_{\alpha}\restriction_{E_m} : E_m \to F$ is linear.

Like in the case of the tangent bundle, by an abuse of notation, a vector bundle (E, M, π) will usually be denoted by E alone. Like for the tangent bundle, E is called the total space or the bundle manifold, M the base manifold, π the bundleprojection and F the typical fiber. For $m \in M$,E_m is called the fiber over m and m is called the base point. The pairs (U_α , X_α) are called local trivializations. A local trivialization (U, X) with $U = M$ is called a global trivialization. If a globaltrivialization exists, the vector bundle is called (globally) trivial.[6]

Theorem (4.4): A vector bundle ξ of rank k is trivial iff there exist sections $\{ \sigma_1 ,\dots, \sigma_k \}$ which are linearly independent at every point of B. [2]

Remark (4.5): Any vector bundle isomorphic to a product bundle is called a trivial vector bundle. [2]

Definition (4.6): E and $\grave{E}$ are vector bundlesover X. Continuous map $f: E \to \grave{E}$ is called a bundle homeomorphism if



"Fig 4"

is commutative and every $f_x: E_x \to \grave{E}_x$ is linear.[12]

5. The Tangent Bundles

Definition (5.1): Let M be a C^k-manifold, let $I \subset R$ be an open interval and let $y: I \to M$ be a C^k-curve for every $t \in I$, the tangent vector $\dot{y}(t)$ of y at t is an element of the tangent space $T_{y(t)}M$. Hence, while t runs through, $\dot{y}(t)$ runs through the tangent spaces along y . To follow the tangent vectors along y it is converint to consider the totality of all tangent spaces of M . This leads to the notion of tangent bundle of a manifold M, denoted by TM . As a set, TM is given by the disjoint of the tangent spaces at all points of M , that is m,

$$TM = \cup_{m \in M} T_m M \qquad (5.1)$$

Let $\pi: TM \to M$ be the canonical projection which assigns to element of $T_m M$ the point m for every $m \in M. TM$ can be equipped with a manifold structure as follows. Denote $n: dimM$. Choose a countable atlas $\{(U_\alpha, k_\alpha): \alpha \in A\}$ on M and define the mapping.

$$k_\alpha^T: \pi^{-1}(U_\alpha) \to R^n \times R^n, k_\alpha^T(X_m) \coloneqq (k_\alpha(m), X_m^{k_\alpha}) \qquad (5.2) \quad [6]$$

Definition (5.2): The coordinate neighborhood N, ψ of q defined in the way is called a normal coordinate neighborhood [14].

i. Tangent Bundles of a Smooth Manifold:

Definition (5.3): For a smooth map *f : X → Y* of manifolds, the induced map $Tf: TX \to TY$ is defined by the formula

[x, i, v] → [y, j, u] ,

where, $(U_i, \emptyset_i), (V_j, \emptyset_j)$ are charts for X, Y respectively at x, y such that

$$f(U_i) \subset V_j, y = f(x), \text{and } u = D(\emptyset_j \circ f \circ \emptyset_i^{-1})_{(v)}. \qquad (5.3)$$

ii. Geometric Tangent Vectors:

Let us define the geometric tangent space to R^n at the point $a \in R^n$, denoted by R_a^n , to be the set $\{a\} \times R^n$. More explicitly,

$$R_a^n = \{(a, v): v \in R^n\} \qquad (5.4)$$

A geometric tangent vector in R^n is an elements of this space. As a matter of notation, we will abbreviate (a, v) as v_a (or sometime v/a if it is clearer, for example if v itself has a subscript). We think of v_a as the vector v with its initial point at a .This set R^n_a is a real vector space (obviously isomorphic to R^n itself) under the natural operations

$$v_a + w_a = (v + w)_a;$$
$$C(v) = c(a). \qquad (5.5) \qquad [7]$$

A tangent vector is the algebraic object that results when the operation of differentiation is performed on a geometric object, namely on a parameterized curve.[4]

Definition (5.4): For every $p \in R^n$, the set of all tangent vectors at p , denoted by $T_p(R^n)$, is called the tangent space to R^n at p [4]

A vector fields are customarily denoted by symbols like $X, Y or Z$, and the vector $X(p)$ is often denoted by X_p (sometimes X may be used to denote a single vector, in some M_p). [11]

iii.Tangent Bundles and Vector Fields:

Let M be a manifold with an atlas $= \{(\emptyset_\alpha, U_\alpha)\}_{\alpha \in A}$. The tangent bundle of M is define as the disjoint union of the tangent spaces. i.e.

$$TM = \cup_{p \in M} \{p\} \times T_p M = \{(p, v) | p \in M, v \in T_p M\} \qquad (5.6)$$

Definition (5.5) Let M be a smooth m-manifold A(smooth) vector field on M is a collection of tangent vectors $\mathrm{X(p)} \in \mathrm{T_pM}$, one for each point $p \in M$, such that the map $M \to TM: p \mapsto (p, X(p))$ is smooth . The set of smooth vector field on M will be denoted by vect(M) .[5]

6.Cross-Section of a Bundle:

Definition (6.1): Let X be a bundle over the base B, with projection $\pi: X \to B$. Then a cross-section of X is a continuous map $k: B \to X$ such that πk= identity.

Example (6.2): A continuous vector field on a manifold is a cross-section of the tangent bundle of the manifold.[8]

i. Intuition behind a Tangent Bundle:

In mechanics, to each nD configuration manifold M there is associated its $2nD$ velocity phase–space manifold, denoted by TM and called the tangent bundle of M. The original smooth manifold M is called the base of TM. There is an onto map $\pi : TM \to M$, called the projection. Above each point $x \in M$ there is a tangent space $T_xM = \pi^{-1}(x)$ to M at x, which is called a fiber. The fiber $T_xM \subset TM$ is the subset of TM, such that the total tangent bundle, $= \cup_{m \in M} T_xM$, is a disjoint union of tangent spaces T_xM to M for all points $x \in M$. From dynamical perspective, the most important quantity in the tangent bundle concept is the smooth map $v : M \to TM$, which is an inverse to the projection π,i.e., $\circ\ v = Id_M$, $(v(x)) = x$. It is called the velocity vector–field. Its graph $(x, v(x))$ represents the cross–section of the tangent bundle TM. This explains the dynamical term velocity phase–space, given to the tangent bundle TM of the manifold M.

7.Command/Control in Human-Robot Interactions:

Suppose that we have ahuman–robot team, consisting of m robots and n humans. To be able to put the modeling of the fully controlled human–robot team performance into the rigorous geometrical settings, we suppose that all possible behaviors of m robots can be described by a set of continuous and smooth, time–dependent robot configuration coordinates $x^r = x^r(t)$,while all robot–related behaviors of n humans can be described by a set of continuous and smooth, time–dependent human configuration coordinates $q^h = q^h(t)$. In other words, all robot coordinates, $x^r = x^r(t)$, constitute the smooth Riemannian manifold M_g^r(such that $r = 1,\dots, dim(M_g^r)$), with the positive–definite metric form

$$g \mapsto ds^2 = g_{rs}(x)dx^r dx^s \quad (7.1)$$

Similarly, all human coordinates $q^h = q^h(t)$, constitute a smooth Riemannian manifold N_a^h (such that $h = 1,\dots,dim(N_a^h)$) , with the positive–definite metric form

$$a \mapsto d\sigma^2 = a_{hk}(q)dq^h\,dq^k. \tag{7.2}$$

In this Riemannian geometry setting, the feed forward command/control action of humans upon robots is defined by a smooth map, $C : N_a^h \to M_g^r$, which is in local coordinates given by a general (nonlinear) functional transformation

$$x^r = x^r(q^h), \left(r = 1,\dots,dim\left(M_g^r\right);\ h = 1,\dots,dim(N_a^h)\right), \tag{7.3}$$

while its inverse, the feedback map from robots to humans is defined by asmooth map, $F = C^{-1} : M_g^r \to N_a^h$ which is in local coordinates given by an inverse functional transformation

$$q^h = q^h(x^r), (h = 1,\dots,dim(N_a^h);\ r = 1,\dots,dim(M_g^r)). \tag{7.4}$$

Now, although the coordinate transformations (7.3) and (7.4) are completely general, nonlinear and even unknown at this stage, there is something known and simple about them: the corresponding transformations of differentials are linear and homogenous, namely

$$dx^r = \frac{\partial x^r}{\partial q^h}dq^h, \quad \text{and} \quad dq^h = \frac{\partial q^h}{\partial x^r}dx^r,$$

which imply linear and homogenous transformations of robot and human velocities,

$$\dot{x}^r = \frac{\partial x^r}{\partial q^h}\dot{q}^h, \text{and}\ \dot{q}^h = \frac{\partial q^h}{\partial x^r}\dot{x}^r \tag{7.5}$$

Relation (7.5), representing two autonomous dynamical systems, given by two sets of ordinary differential equations (O.D.Es.), geometrically defines two velocity vector–fields: (i) robot velocity vector–field, $v^r \equiv v^r(x^r,t) := \dot{x}^r(x^r,t)$; and (ii) human velocity vector–field, $u^h \equiv u^h(q^h,t) := \dot{q}^h(q^h,t)$. Recall that a vector–field defines a single vector at each point x^r (in some domain U) of a manifold in case. Its solution gives the flow, consisting of integral curves of the vector–field, such that all the vectors from the vector–field are tangent to integral curves at different points $xI \in U$. Geometrically, a velocity vector–field is defined as a cross–section of the tangent bundle of the manifold. In our case, the robot velocity vector–

field$v^r = \dot{x}^r(x^r, t)$represents a cross–section of the robot tangent bundleTM_g^r, while the human velocity vector field$u^h = \dot{q}^h(q^h, t)$, represents a cross–section of the human tangent bundleTN_a^h. In this way, two local velocity vector–fields,v^randu^h , give local representations for the following two global tangent maps,$TC : TN_a^h \to TM_g^r$,and$TF : TM_g^r \to TN_a^h$

To be able to proceed along the geometry dynamical line, we need next to formulate the two corresponding acceleration vector–fields,$a^r \equiv a^r(x^r, \dot{x}^r, t)andw^h \equiv w^h(q^h, \dot{q}^h, t)$,as time rates of change of the two velocity vector–fieldsv^randu^h. Now, recall that the acceleration vector–field is defined as the absolute time derivative,$\dot{\bar{v}}^r = \frac{D}{dt} v^r$, of the velocity vector–field. In our case, we have the robotic acceleration vector–field$a^r := \dot{\bar{v}}^r$defined on$M_g^r$by

$$a^r := \dot{\bar{v}}^r = \dot{v}^r + \Gamma_{st}^r v^s v^t = \ddot{x}^r + \Gamma_{st}^r \dot{x}^s \dot{x}^t, \quad (7.6)$$

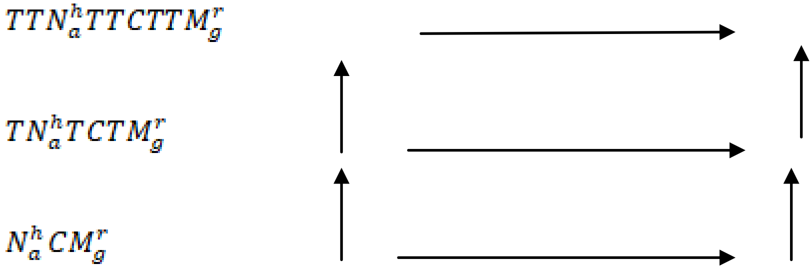
and the human acceleration vector–fieldw^h:=u^h defined on$N_a^h$by

$$w^h := \dot{\bar{u}}^h = \dot{u}^h + \Gamma_{jk}^h u^j u^k = \ddot{p}^h + \Gamma_{jk}^h \dot{q}^j \dot{q}^k \quad (7.7)$$

Geometrically, an acceleration vector–field is defined as a cross–section ofthe second tangent bundle of the manifold. In our case, the robot acceleration vector–field$a^r = \dot{\bar{v}}^r(x^r, \dot{x}^r, t)$,given by the O.D.Es (7.6), represents across–section of the second robot tangent bundle TTM_g^r, while the human acceleration vector–fieldw^h=$\dot{\bar{u}}^h(q^h, \dot{q}^h, t)$,given by the ODEs (7.7), represents a cross–section of the second human tangent bundleTTN_a^h. In this way, two local acceleration vector–fields,a^randw^h , give local representations for the following two second tangent maps

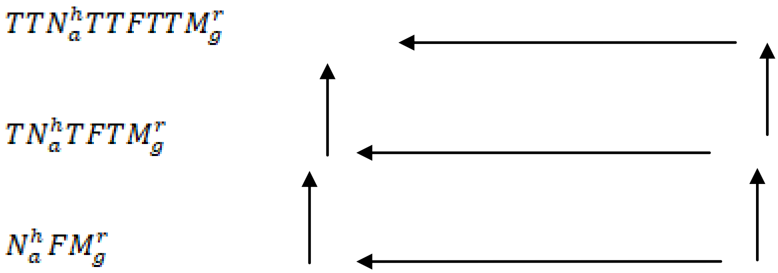
$$TTC : TTN_a^h \to TTM_g^r, \text{ and } TTF : TTM_g^r \to TTN_a^h.$$

In other words, we have the feed forward command/control commutative diagram:



"Fig 5"

As well as the feedback commutative diagram:



"Fig 6"

These two commutative diagrams formally define the global feed forward and feedback human–robot interactions at the positional, velocity, and acceleration levels of command and control.[13]

8.Results

It is involved in many engineering sciences, modern electronic industries, technical sciences, artificial intelligence and modern machine industry, for example the robotics industry, which performs most of the movement of the human dynamically and operates mainly on the basis of the tangent bundles, enables human to make memory work as a neural cell alternative to the human brain. The study also concluded: The concept of the tangent space as a basic concept of the tangent bundles**,**we came to the concept of fiber bundles from a general perspective**,** that tangent bundle have a great role in many engineering fields.

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