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On Common Fixed Point for Four Maps in 2- metric Space

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Abstract:

We have established a fixed point theorem in 2-metric space for four maps. Our result generalizes the result of Lal and Singh.

Introduction:

There have been a number of generalization of a metric space. One such generalization of 2-metric space was initiated by Gahler [1]. Geometrically in plane, 2-metric function abstracts the properties of the area function for Euclidean triangle just as a metric function abstracts the length function for Euclidean segment.

After the introduction of concept of 2-metric space, many authors established an analogue of Banach’s Contraction principle in 2-metric space. Iseki [2] for the first time developed fixed point theorem in 2-metric space. Since then a quite significant number of authors [3], [4], [5], [6], etc. have established fixed point theorem in 2- metric space.

Lal and Singh[3] proved,

Theorem (1.1) Let S and T are two self maps of a complete 2 metric space (X, d) such that:

$$d(Sx, Ty, a) \le a_1 d(x, y, a) + a_2 d(Sx, x, a) + a_3 d(Ty, y,a) + a_4 d(Sx, y,a) + a_5 d(Ty, x, a)$$

for all x, y, a $\in$X, where a_i (i=1, 2, 3, 4,5) are positive integers such that $(1-a_3-a_4) >0$ and $(1-a_2-a_5) >0$.

Then S and T have a unique fixed point theorem.

Preliminaries: Now we give some basic definitions and well known results that are needed in the sequel.

Definition (2.1) [1] Let X be a non-empty set and d: X xXxX → R+. If for all x, y, z, and u in X. We have

(d_1) d(x, y, z) = 0 if at least two of x, y, z are equal.

(d_2) for all $x \neq y$, there exists a point z in x such that $d(x, y, z) \neq 0$.

(d_3) d(x, y, z) = d(x, z, y) = d(y, z, x) =.............. and so on

(d_4) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$

Then d is called a 2-metric on X and the pair (X, d) is called 2-metric space.

Definition (2.2): A sequence $\{x_n\}_{n \in N}$ in a 2-metric space (X,d) is said to be a cauchy sequence if $\lim_{\substack{n\to\infty \\ m\to\infty}} d(x_n, x_m, a) = 0$ for all a ϵ X.

Definition (2.3): A sequence $\{x_n\}_{n \in N}$ in a 2-metric space (X,d) is said to be a convergent if $\lim_{n\to\infty} d(x_n, x, a)$ = 0 for all a ϵ X. The point x is called the limit of the sequence.

Definition (2.4) : A 2-metric space (X,d) is said to be complete if every cauchy sequence in X is convergent.

Main Result:

Theorem (3.1) : Let A, B, S and T are four self maps of a complete 2-metric space (X, d) such that

(i) $A(X) \subseteq T(X) : B(X) \subseteq S(X)$

(ii) pairs (A,S) and (B, T) are commutating.

(iii) $d(Ax, By, a) \leq a_1 d(Sx, Ty, a) + a_2 d(Ax, Sx, a) + a_3 d(By, Ty, a) + a_4 d(Sx, By, a) + a_5 d(Ax, Ty, a)$

for all x, y, a $\in$X, where a_i(i=1, 2, 3, 4) are positive integers such that

and $(1-a_3-a_4) > 0$ and $(1-a_2-a_5) > 0$

then

(iv) A and S have a coincidence point

(v) B and T have a coincidence point

Moreover if the pairs (A, S) and (B, T) are commutating then A,B,S and T have a unique common fixed point.

Proof :

Since (i) holds, we can define a sequence by choosing an arbitrary point x_0 in X, such that

$X_{2n} = Ax_{2n} = Tx_{2n+1}$

and $x_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$ for n = 0, 1, 2.....

Now first we prove that $d(x_{2n}, x_{2n+1}, x_{2n+2}) = 0$

$$d(x_{2n}, x_{2n+1}, x_{2n+2}) = d(Bx_{2n+1}.Ax_{2n+2}, x_{2n})$$
$$= d(Ax_{2n+2}, Bx_{2n+1}, x_{2n})$$
$$\leq a_1 d(Sx_{2n+2}, Tx_{2n+1}, x_{2n}) + a_2 d(Ax_{2n+2}, x_{2n+2}, x_{2n})$$

$+ a_3 d(Bx_{2n+1}, Tx_{2n+1}, x_{2n}) + a_4 d(Sx_{2n+2}, Bx_{2n+2}, x_{2n}) +$
$a_5 d(Ax2_{n+2}, Tx2_{n+1}, X_n)$
$= a_1 d(x_{2n+1}, x_{2n}, x_{2n}) + a_2 d(x_{2n+2}, x_{2n+1}, x_{2n})$
$+ a_3 d(x_{2n+1}, x_{2n}, x_{2n}) + a_4 d(x_{2n+1}, x_{2n+1}, x_{2n}) + a_5 d(x_{2n+2}, x_{2n}, x_{2n})$

i.e. $(1 - a_2)\, d(x_{2n}, x_{2n+1}, x_{2n+2}) \leq 0$. which is a centradiction.

Hence $d(x_{2n}, x_{2n+1}, x_{2n+2}) = 0$

Now we shall prove that $\{x_n\}$ is cauchy sequence in X. For this we put $x = x_{2n}$, $y = x_{2n+1}$ in (iii), we get

$d(Ax_{2n}, x_{2n+1}, a) = d(Ax_{2n}, Bx_{2n+1}, a)$
$\leq a_1 d(Sx_{2n}, Tx_{2n+1}, a) + a_2 d(Ax_{2n}, Sx_{2n}, a) + a_3 d(Bx_{2n+1}, Tx_{2n+1}, a)$
$+ a_4 d(Sx_{2n}, Bx_{2n+1}, a) + a_5 d(Ax_{2n}, Tx_{2n+1}, a)$
$= a_1 d(x_{2n-1}, x_{2n}, a) + a_2 d(x_{2n}, x_{2n-1}, a) + a_3 d(x_{2n+1}, x_{2n}, a)$
$+ a_4 d(x_{2n-1}, x_{2n+1}, a) + a_5 d(x_{2n}, x_{2n}, a)$
$\leq (a_1 + a_2) d(x_{2n-1}, x_{2n}, a) + a_3 d(x_{2n+1}, x_{2n}, a)$
$+ a_4 [d(x_{2n-1}, x_{2n+1}, x_{2n}) + d(x_{2n-1}, x_{2n}, a) + d(x_{2n}, x_{2n+1}, a)] +$

i.e. $d(x_{2n}, x_{2n+1}, a) \leq (a_1 + a_2 + a_4)\, d(x_{2n-1}, x_{2n}, a) + (a_3 + a_4)\, d(x_{2n}, x_{2n+1}, a)$
or, $d(x_{2n}, x_{2n+1}, a) \quad d(x_{2n-1}, x_{2n}, a)$.
or $d(x_{2n}, x_{2n+1}, a) \leq \gamma d(x_{2n-1}, x_{2n}, a)$ where
Again putting $x = x_{2n+2}$, $y = x_{2n+1}$ (iii) we get, $y = x_{2n+1}$,

$d(x_{2n+1}, x_{2n+2}, a) = d(Bx_{2n+1}, Ax_{2n+2}, a)$
$= d(Ax_{2n+2}, Bx_{2n+1}, a)$
$\leq a_1 d(Sx_{2n+2}, Tx_{2n+1}, a) + a_2 d(Ax_{2n+2}, Sx_{2n+2}, a) + a_3 d(Bx_{2n+1}, Tx_{2n+1}, a)$
$+ a_4 d(Sx_{2n+2}, Bx_{2n+1} a) + a_5 d(Ax_{2n+2}, Tx_{2n+1}, a)$.
$= a_1 d(x_{2n+1}, x_{2n}, a) + a_2 d(x_{2n+2}, x_{2n+1}, x_{2n+1}, a) + a_3 d(x_{2n+1}, x_{2n}, a)$
$+ a_4 d(x_{2n+1}, x_{2n+1}, a) + a_5 d(x_{2n+2}, x_{2n}, a)$
$\leq (a_1 + a_3)\, d(x_{2n+1}, x_{2n}, a) + a_2 d(x_{2n+2}, x_{2n+1}, a)$
$+ a_5 (d(x_{2n+2}, x_{2n}, x_{2n+1}) + d(x_{2n+2}, x_{2n+1}, a) + d(x_{2n+1}, x_{2n}, a)]$.

= $\quad (a_1 + a_2 e a_5)\, d(x_{2n+1}, x_{2n}, a) + (a_2 + a_5)\, d(x_{2n+2}, x_{2n+1}, a)$
or $\quad d(x_{2n+1}, x_{2n+2}, a) \quad d(x_{2n+1}, x_{2n}, a)$
or $\quad d(x_{2n+1}, x_{2n+2}, a) \leq \beta d(x_{2n+1}, x_{2n}, a)$, where
$\leq \beta.\gamma\, d(x_{2n-1}, x_{2n}, a)$
⋮
$\leq d(\beta\gamma)^n\, d(xo, x_1, a)$.

Let $c = \beta\gamma$, then $d(x_{2n+1}, x_{2n+2}a)\ c^n d(x_o, x, a)$, where $0 \le c < 1$.
Hence $\{x_n\}$ is a cauchy sequence. Since (X, d) is a complete 2-metric space,
$\{x_n\}$ converges say to z. Hence the sequence $Ax_{2n} = Tx_{2n+1}$ and $Bx_{2n+1} = x_{2n+2}$ which are subsequence also converge to point z.
Since $B(X) \le S(X)$, there exists a point $u \in X$ st. $z = Su$.

Now $d(Au, z, a) = d(Au, Bx_{2n+1}, a)$

$\le a_1 d(Su, Tx_{2n+1}, a) + a_2 d(Au, Su, a) + a_3 d(Bx_{2n+1}, Tx_{2n+1}, a) + a_4 d(Su, Bx_{2n+1}, a) + a_5 d(Au, Tx_{2n+1}, a)$

when $n \to$, $Tx_{2n+1} \to z$, $Bx_{2n+1} \to z$ and putting $Su = z$.
$d(Au, z, a) \le (a_2 + a_5) d(Au, z, a)$, which is a contradiction.
Hence, $d(Au, z, a) = 0$ which gives $Au = z$.
Thus, $Su = Au = z$.
So, u is the coincidence point of A and S. Since the pair of maps A and S are commutative, ASu -= SAu i.e. Az = Sz.
Again since $A(x) \le T(x)$, there exists a point $v \in X$ s.t. $z = T\nu$.
Now, $d(z, \beta v, a) = d(Ax_{2n}, Bv, a)$

$\le a_1 d(Sx_{2n}, T\nu, a) + a_2 d(Ax_{2n}, Sx_{2n}, a) + a_3 d(B\nu, T\nu, a)$
$a_4 d(Sx_{2n}, Bv, a) + a_5 d(Ax_{2n}, Tv, a)$

When $n \to$ and putting Tv = z., we have
$d(z, Bv, a) \le (a_3 + a_4) d(Bv, z, a)$, which is a contradiction.
Thus, $d(z, Bv, a) = 0$ which implies z = Bv.
i.e. z = Tv, = Bv, showing that v is a coincidence point of T and B. As the pair of maps B and T are commutative
so that TBv = BTv i.e. Tz= Bz.

Now we shall show that z is a fixed point of A.
$d(z, Az, a) = d(Az, Bv, a)$

$\le a_1\ d(Sz, Tv, a) + a_2 d(Az, Sz, a) + a_3 d(Bv, Tv, a) + a_4 d(Sz, BV, a) + a_5 d(Az, Tv, a)$

$= a_1 d(Az, z, a) + a_2 d(Az, Az, a) + a_3 d(z, z, a) + a_4 d(Az, z, a) + a_5 d(Az, z, a)$

$\therefore \quad d(z, Az, a) \le (a_1 + a_4 + a_5)\ d(Az, Az, a)$ which is not possible.

Therefore, $d(z, Az, a) = 0$ which given Az = z. i.e. Az = Sz = z.

Now we shall show that z is a fixed point of B.
$d(z, Bz, a) = d(Az, Bz, a)$

$\le a_1 d(Sz, Tz, a) + a_2 d(Az, Sz, a) + a_3 d(Bz, Tz, a) + a_4 d(Sz, Bz, a) + a_5 d(Az, Tz, a)$

$= a_1 d(z, Bz, a) + a_2 d(z, z, a) + a_3 d(Bz, Bz, a) + a_4 d(z, Bz, a) + a_5 d(z, Bz, a)$

or $d(z, Bz, a) \le (a_1 + a_4 + a_5)\ d(z, Bz, a)$ which is a contradiction.
Thus, $d(z, Bz, a) = 0$ which gives z = Bz. i.e. z = Bz = Tz.
Hence, Az = Sz = Bz = Tz = z, showing that z is the common fixed point of A, B, S, and T.

Now we shall show that the uniqueness of this fixed point.
Suppose that w is the other common fixed point of A, B, S and T.
Then we have Aw= Sw = BW = Tw = w.
Now,

$$d(z, w, a) = d(Az, Bw, a)$$
$$= a_1 d(Sz, Tw, a) + a_2 d(Az,Sz, a) + a_3 d(Bw, Tw,a)$$
$$+ a_4 d(Sz,Bw,a) + a_5 d(Az, Tw, a)$$
$$= a_1 d(z, w, a) + a_2 d(z, z, a) + a_3 d(w,w,w) + a_4 d(z,w,a) + a_5 d(z, w, a)$$

i.e. $d(z, w,a) \le (a_1 + a_4 + a_5)\, d(z, w, a)$ which is a contradiction
hence $d(z, w, a) = 0$ which gives $z = w$.
i.e. our supposition is wrong and therefore z is the unique common fixed point of A, B, Sand T.//

Remark (3.2): By putting A=T and S=B we get the theorem 1.1. Thus our result generalizes the result of Lal and Singh[3].

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