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# Construction of A Homogeneous Factorization of the Hamming Graph $H_1(n,q)$ Using the Imprimitive Action of $N \leq S_q$ on the Arc Set of The Complete Graph $K_q$

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**Abstract:**

*If $\Gamma = (V(\Gamma), A(\Gamma))$ is a digraph with vertex set $V(\Gamma)$ and arc set $A(\Gamma)$, then a homogeneous factorization of $\Gamma$ of index n is the 4 – tuple $(M, G, V(\Gamma), P)$ such that*

1. $P = \{P_1,\ldots,P_n\}$ *is a partition of* $A(\Gamma)$.
2. $G \leq Aut(\Gamma)$ *acting transitively on P.*
3. *M is the kernel of the action of G and is transitive on* $V(\Gamma)$.

*In this paper, it will be shown that there exists a unique homogeneous factorization $(M, G, V(\Gamma), P)$ of index n where $\Gamma = H_1(n,q) = (K_q)^n$, using the imprimtive action of $N \leq S_q$ on $A(K_q)$. This factorization satisfies the following:*

1. $M = N^n$
2. $G = M \times T$, *where T is an abelian subgroup of* $S_n$ *acting regularly on* $\Omega = \{1, 2, \ldots, n\}$.
3. $P = \{P_1,\ldots,P_n\}$, *such that*
   a. $P_j = \bigcup_{i=1}^{r} P_{ij},\ 1 \leq j \leq n$
   b. $P_{ij} = \{((u_1,\ldots,u_j,\ldots,u_n),(u_1,\ldots,v_j,\ldots,u_n)) \mid (u_j,v_j) \in B_i\}$ *where* $((u_1,\ldots,u_j,\ldots,u_n),(u_1,\ldots,v_j,\ldots,u_n)) \in A((K_q)^n)$

*and* $B = \{B_1, \ldots, B_r\}$ *is a system of imprimitive blocks.*

**Key words:** Hamming graph, homogeneous factorization, graph product, permutation group, group action, transitive group action.

## I. Introduction

If $G = ((V(\Gamma), A(\Gamma))$ is a digraph with vertex set $V(\Gamma)$ and arc set $A(\Gamma)$, a *homogeneous factorization* of $\Gamma$ of index $n$ is a partition $P = \{P_1, \ldots, P_n\}$ of $A(\Gamma)$ such that there exists a subgroup $G \leq Aut(\Gamma)$ that leaves $P$ invariant, permutes the elements of $P$ transitively, and the kernel $M$ of the action is transitive on $V(\Gamma)$. The group G induces an isomorphism between each pair of subgraphs $(V(\Gamma), P_i)$. The symbol $(M, G, V(\Gamma), P)$ is used to mean the homogeneous factorization of $\Gamma$ when it is necessary to emphasize the underlying groups. The subgraphs $((V(\Gamma), P_i)$ are known as the *factors* of the homogeneous factorization.

In a paper by [Guidici, *et. al.*, 2004], methods of constructing homogeneous factorization of digraphs were introduced. One of the methods given is that of the homogeneous factorization of index $n$ of a given graph $\Sigma = (\Gamma)^n$, where $\Gamma$ is a digraph. In the construction, a primitive action of a subgroup $N \leq S_q$ on the arc set of $\Gamma$ is assumed. The $n$ orbits of $N$ on $A(\Gamma)$ were used in defining the cells of the partitions of $A(\Sigma)$. The above construction served as a motivation in this paper. In this paper, we take $\Gamma = K_q$. Specifically, we want to answer the following questions:

Is it possible to construct a homogeneous factorization of $(K_q)^n$ of some index k using the imprimitive action of a subgroup of $S_n$?

## II. Theoretical Necessities

Knowledge of elementary graph theory and permutation groups shall be assumed in this paper. However, some definitions will still be given in order to better understand this study.

### 2.1. Graph Theoretic Necessities

A *complete graph* $K_q$ is a graph with $q$ vertices such that each pair of distinct vertices are adjacent to each other. The $n^{th}$ *Cartesian* product of any graph $\Gamma$ with itself is the graph $\Gamma^{\square n}$, whose vertex set $V(\Gamma^{\square n})$ is the Cartesian product of $n$ copies of $V(\Gamma)$. Now, two vertices $(u_1,...,u_j,...,u_n)$ and $(u_1,...,v_j,...,u_n)$ are adjacent iff $(u_i,u_j) \in A(\Gamma)$ for all $u_i, v_j \in V(\Gamma)$. The $n^{th}$ *Cartesian* product of $K_q$ with itself is the graph $H_1(n,q) = (K_q)^n$ known as the *Hamming graph.* The cube is the Hamming graph $H_1(3,2)$. In general, the cubic graph $Q_n$ is the Hamming graph $H_1(n,2)$.

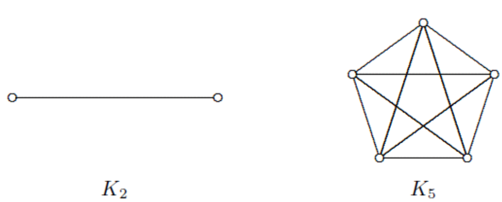


**Figure 1. Complete Graphs $K_2$ and $K_5$**

In this study, we shall be concerned with the homogeneous factorization of the Hamming graph $H_1(n,q)$ for $q \geq 3$.

Two graphs are $\Gamma_1$ and $\Gamma_2$ are *isomorphic* if there exists a bijection $\phi: V(\Gamma_1) \to V(\Gamma_2)$ such that $(u,v) \in A(\Gamma_1)$ iff $(\phi(u), \phi(v)) \in A(\Gamma_2)$. A *graph automorphism* or simply an *automorphism* is an isomorphism of a graph with itself. The set of all automorphisms of $\Gamma$ form a group under function

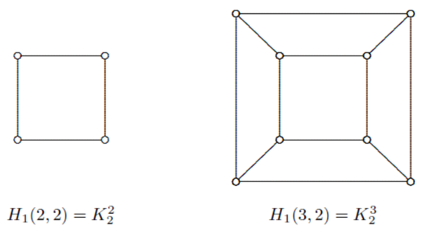


**Figure 2 The Hamming Graphs $H_1(2,2)$ and $H_1(3,2)$**

composition. This is known as the automorphism group of $\Gamma$ and is denoted by *Aut*($\Gamma$). For the complete graph $K_q$. $Aut(K_q) = S_q$ [Biggs, 1974].

## 2.2. Concepts from Permutation Group Theory

A *permutation group G* on a non empty set $\Omega$ is a subgroup of $Sym(\Omega)$. The image of the action of $G$ on $\Omega$ is a permutation group, called the permutation group *induced* on $\Omega$ by G. Also, $\Omega$ is known as a *G-space.*

The *kernel* of the action is the set $M = \{x \mid \alpha^x = \alpha, \forall \alpha \in \Omega\}$. When the kernel contains only the identity element, then we say that the action is *faithful.*

*Example 2.2.1*. Every subgroup $G$ of $Sym(\Omega)$ acts naturally on $\Omega$ where $\alpha^g$ refers to the image of $\alpha$ under the permutation $g \in G$.

Example 2.2.2. A natural action of $S_q$ on $V(K_q)$ is given by $(u_i)^g = u_{i^g}$, for $u_i \in V(K_q)$, $g \in S_q$ and $i^g$ is the image of $i$ under the permutation $g$.

If $\Omega$ is a *G-space*, then the *orbit* of $\alpha \in \Omega$ under $G$ is the set $\alpha^G = \{\alpha^x \mid x \in G\}$, while the *stabilizer* of $\alpha$ is the set $G_\alpha = \{x \in G \mid \alpha^x = \alpha\}$.

If for all $\alpha \in \Omega$, $\alpha^G = \Omega$, then we say that the action of $G$ is *transitive*. Consequently, if $\alpha$ and $\beta$ are in $\Omega$, then there exists a $g \in G$ such that $\alpha^g = \beta$. A transitive action that is both transitive and faithful is called *regular*.

Let $\Omega$ be a $G$ – *space*, and $\phi \neq \Delta \subseteq \Omega$, such that $\Delta^x := \{\delta^x \mid \delta \in \Delta\}$. If $G$ is transitive on $\Omega$, then $\Delta$ is a *block* of $\Omega$ for $G$ if either $\Delta^x = \Delta$ or $\Delta^x \cap \Delta = \phi$. $\Omega$ and $\{\beta\}$ are *trivial* blocks of $\Omega$ respectively, for any $\beta \in \Omega$. Any other blocks are known as *nontrivial* blocks. The action of $G$ on $\Omega$ is *primitive* if it has no nontrivial blocks on $\Omega$. Otherwise, we say that the action is *imprimitive*.

If $\Delta$ is a block of $\Omega$, then the set $\mathcal{D} = \{\Delta^g \mid g \in G\}$ is a *system of blocks* containing $\Delta$.

A group $N \in Aut(\Gamma)$ is said to be *arc transitive* on $\Gamma$ iff $N$ acts transitively on $A(\Gamma)$.

The following Lemma was proven in [Rosal, 2004]. This Lemma would be most useful in our construction.

*Lemma* 2.2.1. There exists a subgroup of $S_q$ acting imprimitively on the arc set $A(K_q)$ of $K_q$, for $q \geq 3$.

*Example* 2.2.3. For $K_3$, let $V(K_3) = \{u_1, u_2, u_3\}$ and $B = \{(u_1, u_2), (u_{2,} u_1)\} \subseteq A(K_3)$. Then $B$ is a nontrivial block of $A(K_3)$. Thus $S_3$ acts imprimitively on $A(K_3)$.

A group $G$ is said to be *arc transitive* on a graph $\Gamma$ if it is transitive on $A(\Gamma)$. Consequently, G is *vertex transitive* on $\Gamma$, that is, it is transitive on $V(\Gamma)$

## III. The Construction

Let $N \leq S_q, q \geq 3$ be imprimitive on $A(K_q)$. Then there exists a nontrivial block $B$ of $A(K_q)$. Let $\mathrm{B} = \{B^g \mid g \in N\}$ be a block system. If $|\mathrm{B}| = r$, then $r \geq 2$ and $r \mid B| = q(q-1)$. Thus, we may write $\mathrm{B} = \{B_1, ..., B_r\}$, where $B_i = B^m$ for some $m \in N$. We now define the following:

$$\Sigma := (K_q)^n = H_1(n, q)$$

$$P_{ij} := \{((u_1, ..., u_j, ..., u_n), (u_1, ..., v_j, ..., u_n)) \\ \mid (u_j, v_j) \in B_i\}$$

where,

$$((u_1, ..., u_j, ..., u_n), (u_1, ..., v_j, ..., u_n)) \in A(\Sigma)$$

$$P_j = \bigcup_{i=1}^{r} P_{ij}$$

$$\mathrm{P} = \{P_1, ..., P_n\}$$

$M := N^n$

$G := M \times T$, where $T \leq S_n$,

T is abelian and acting regularly on $\Omega = \{1, 2, ...n\}$.

We use the above sets to construct the needed homogeneous factorization. To start we introduce the following lemmas. Some proofs will be omitted.

*Lemma* 3.1 $\mathrm{P}$ partitions $A(\Sigma)$.

*Lemma* 3.2. $M$ is transitive on $V(\Sigma)$ and fixes $P_i, \forall i = 1, 2, ..., n$, setwise.

*Proof*: Let $m = (m_1, ..., m_n) \in M$, and let $u = (u_1, ..., u_n) \in V(K_q)$. Then $M$ acts on $V(\Sigma)$ via the map $u^m = (u_1^{m_1}, ..., u_n^{m_n})$.

If $(u_1, ..., u_n)$ and $(v_1, ..., v_n)$ are in $V(\Sigma)$, then $\forall i \in \Omega$, $\exists g_i \in N$ such that $u_i^{g_i} = v_i$. This is so since $N$ is transitive on $V(\Sigma)$. Take $g = (g_1, ..., g_n) \in M$, then $(u_1, ..., u_n)^g = (u_1^{g_1}, ..., u_n^{g_n}) = (v_1, ..., v_n)$. Thus, $M$ is transitive on $V(\Sigma)$. If $((u_1, ..., u_j, ..., u_n), (u_1, ..., v_j, ..., u_n)) \in A(\Sigma)$, then $((u_1, ..., u_j, ..., u_n), (u_1, ..., v_j, ..., u_n)) \in P_j$ for some $j \in \Omega$. Consequently, there exists a $B_i \in \mathrm{B}$ such that $(u_j, v_j) \in B_i$. If, $g = (g_1, ..., g_n)$ is in $M$ then, $((u_1, ..., u_j, ..., u_n), (u_1, ..., v_j, ..., u_n))^g = ((u_1, ..., u_j, ..., u_n)^g, (u_1, ..., v_j, ..., u_n)^g) = ((u_1^{g_1}, ..., u_j^{g_j}, ..., u_n^{g_n}), (u_1^{g_1}, ..., v_j^{g_j}, ..., u_n^{g_n}))$. Since each $g_i \in N$, then $u_k^{g_k} \in V(K_q)$. Also, since $(u_j, v_j) \in B_i$, then

$(u_j, v_j)^{g_j} = (u_j^{g_j}, v_j^{g_j}) \in (B_i)^{g_j} = B_t$, for some $t = 1,...,r$. Hence, $((u_1,...,u_j,...,u_n),(u_1,...,v_j,...,u_n))^g \in P_{tj} \subseteq P_j$. Therefore, $M$ fixes $P_j$ setwise.

□□□

*Lemma* 3.3. $T$ acts

1. naturally on $V(\Sigma)$ via the map $(u_1,...,u_n)^f = (u_{1^f},...,u_{n^f})$.
2. on $M$ via the map $(m_1,...,m_n)^f = (m_{1^f},...,m_{n^f})$

for all $(u_1,...,u_n) \in V(\Sigma)$ and $(m_1,...,m_n) \in M$.

The proofs of the above Lemmas are straight-forward and will be left to the readers to do.

*Lemma* 3. 4. $M$ and $T$ are subgroups of $Aut(\Sigma)$.

*Proof:* We have already shown that both $M$ and $T$ permute the elements of $V(\Sigma)$. It suffices to show that they preserve adjacency.

For $M$, let $(u,v) \in A(\Sigma)$. Then $(u,v) = ((u_1,...,u_j,...,u_n),(u_1,...,v_j,...,u_n))$, where $u_i = v_i$ whenever $i \neq j$, and $u_j \neq v_j$.

If $m = (m_1,...m_j,...,m_n)$, then $(u,v)^m = ((u_1^{m_1},...,u_j^{m_j},...,u_n^{m_n}),$
$(u_1^{m_1},...,v_j^{m_j},...,u_n^{m_n}))$.

Since $u_j \neq v_j$, then $u_j^{m_j} \neq v_j^{m_j}$ since $m_j$ is a bijection. Thus, $(u,v)^m = (u^m, v^m) \in A(\Sigma)$.

Let $f \in T$, then $(u^f, v^f) = (u,v)^f = ((u_{1^f}, \ldots u_{j^f}, u_{n^f}), (u_{1^f}, \ldots v_{j^f}, u_{n^f}))$.

Since $f \in T$, then $\exists | k \in \Omega$ such that $j = k^f$.

It follows that the $k^{th}$ coordinate of $(u,v)^f$ is $(u_{k^f}, v_{k^f}) = (u_j, v_j) \in A(K_q)$. Hence $(u,v)^f \in A(\Sigma)$. Therefore $M, T \leq Aut(\Sigma)$.

□□□

*Lemma* 3. 5. Let $G = M \times T$. Then $G$ is a group under the binary operation

$$(m, f)(a,b) = (ma^{f^{-1}}, fb)$$

Furthermore, $G$ is a semidirect product of $M$ and $T$.

*Proof*: It is easy to prove that G is a group under the given binary operation. Thus the proof for this shall be omitted. To prove that G is a semi-direct product of $M$ and $T$, we have to show $T$ normalizes $M$ and that $M \cap T = \{1\}$, where 1 is the identity in $G$. We first define the following sets.

Let
$M^* = \{(m,1) \mid m \in M\}$ and $T^* = \{(1, f) \mid f \in T\}$ where 1 is the identity permutation.

Clearly, $M^*$ and $T^*$ are subgroups of $G$ and that $M^* \cong M$ and $T^* \cong T$. Also, $M^* \cap T^* = \{1\}$, where $1 = (1,1)$. Consequently, $M \cap T = \{1\}$

Now, $T^*$ acts on $M^*$ by conjugation: $(m,1)^{(1,f)} = (1, f)^{-1}(m,1)(1, f) = (m^f, 1)$

Consequently, $T^*$ normalizes $M^*$. Since $T^* \cong T$ and $M^* \cong M$, then it follows that $T$ normalizes $M$. Therefore, $G$ is a semidirect product of $M$ and $T$.

□□□

Notice that if $m \in M$ and $f \in T$, then $(m,1)(1,f) = (m,f)$ and $fm = m^{f^{-1}} f$. Obviously, $G$ is not abelian.

Since $M$ is normal in $G = M \times T$, then $G = MT$. For all $m \in M$ and $f \in T$, $fm = f(f^{-1}\hat{m}f) = \hat{m}f$, for some $\hat{m} \in M$. Consequently, $G = TM$.

The following is a direct consequence of the preceding lemmas.

*Lemma* 3. 6. $G \leq Aut(\Sigma)$.

It is easy to show that G acts on $V(\Sigma)$ through the map $u^{mf} = (u^f)^{m^f}$, where $m \in M$ and $f \in T$. Also, since $m, f \in G$, then $(u^m)^f = u^{mf}$.Hence, $(u^m)^f = (u^f)^{m^f}$.

We next show that $G$ acts on $\mathrm{P}$ transitively.

*Lemma* 3. 7. $\mathrm{P}$ is $G$ invariant.

*Proof*: Let $P_j \in \mathrm{P}$ and $(u,v) = ((u_1,...,u_j,...,u_n),(u_1,...,u_j,...,u_n)) \in P_j$. Then for some $r \in \Omega$, $(u,v) \in P_{rj}$. Thus, $(u_j, v_j) \in B_r$. Suppose $g = mf \in G$, then $m = (m_1,...,m_n) \in M$, $m_i \in N$ for each $i \in \Omega$ and $f \in T$. We now compute for $(u,v)^g$.

Now, $(u,v)^m \in P_j$ since $M$ fixes $\mathrm{P}$ setwise from Lemma 3.3. Specifically, $(u,v)^m \in P_{sj}$ for some $s \in \Omega$.

Next we consider $((u,v)^m)^f$.

$((u,v)^m)^f = ((u_{1^{m_1}},..,u_{j^{m_j}},..u_{n^{m_n}}),(u_{1^{m_1}},..,v_{j^{m_j}},..u_{n^{m_n}}))$

Since $f \in T$, just as in the proof of Lemma 3.1.4, there is a unique $k \in \Omega$ such that the $k^{th}$ coordinate of $(u,v)^g = ((u,v)^m)^f = (u_j^{m_j}, v_j^{m_j})$ is in $B_s$. Thus, $(u,v) \in P_{sk} \subseteq P_k \in \mathrm{P}$. Therefore $\mathrm{P}$ is $G$ invariant.

□□□

Finally, we are going to show that the action of $G$ on $\mathrm{P}$ is transitive and the kernel of the action is actually *M*.

*Lemma* 3.8. $G$ acts on $\mathrm{P}$ transitively via the map $P_j^{\,g} = P_{j^{f^{-1}}}$, where $g = mf$. This action is transitive and the kernel of this action is *M*.

*Proof*: From Lemma 3.7., it has been shown that $\mathrm{P}$ is invariant in *G*. Now, $(P_j)^1 = P_{j^1} = P_j$. Also, if $g,h \in G$, then $g = mf$ and $h = ab$, for some $m,a \in M$ and $f,b \in T$. Thus, $((P_j)^g)^h = (P_{j^{f^{-1}}})^h = P_{(j^{f^{-1}})^{b^{-1}}} = P_{j^{(f^{-1}b^{-1})}} = P_{j^{(bf)^{-1}}}$.

On the other hand, $(P_j)^{gh} = (P_j)^{mfab} = (P_j)^{m\hat{a}fb} = (P_j)^{(m\hat{a})fb} = P_{j^{(fb)^{-1}}}$.

Thus, $(P_j)^{gh} = ((P_j)^g)^h$. Also, let $P_j, P_k \in \mathrm{P}$. Since $T$ acts on $\Omega$ transitively, then there exists $x \in T$ such that $j^{x^{-1}} = k$. Taking $h = mx \in G$, for any $m \in M$, then $(P_{j^{x^{-1}}}) = P_k$.

Hence $G$ acts on $\mathrm{P}$ transitively.

Lastly, we show that M is the kernel of the action. Suppose $K = \{g \in G \mid (P_j)^g = P_j\}$ is the kernel. Clearly, $M \subseteq K$. Let $g = mf \in K$, then $P_j^{\,g} = P_{j^{f^{-1}}} = P_j$.

Thus, $j^{f^{-1}} = j$. Therefore, $f^{-1}$ or $f$ is in the kernel of the action of $T$ on $\Omega$. Since the action of is faithful, it follows that $f = 1$. Thus, $g \in M$. Therefore, $K \subseteq M$. Consequently, $K = M$.

□□□

## IV. The Main Result

Since $G \leq Aut(\Sigma)$, then every element $g \in G$ is an automorphism of $\Sigma$. Hence, for any $i, j \in \Omega$, the subgraphs $(V(\Sigma), P_i)$ and $(V(\Sigma), P_j)$ are isomorphic.

Combining all the Lemmas above, we have now proven the main result of this study.

*Proposition* 4.1 There exists a homogeneous factorization $(M, G, V(\Sigma), \mathrm{P})$ of index $n$ of the Hamming graph $\Sigma := H_1(n,q)$, for $q \geq 3$, satisfying the following:

1. $\mathrm{P} = \{P_1, ..., P_n\}$

$$P_j = \bigcup_{i=1}^{r} P_{ij}$$

$$P_{ij} := \{((u_1, ..., u_j, ..., u_n), (u_1, ..., v_j, ..., u_n)) \\ \mid (u_j, v_j) \in B_i\}$$

such that

$$((u_1, ..., u_j, ..., u_n), (u_1, ..., v_j, ..., u_n)) \in A(\Sigma)$$

2. $G := M \times T$, where

   $M := N^n$ and $T \leq S_n$,

   T is abelian and acting regularly on $\Omega = \{1, 2, ...n\}$.

We shall call the above factorization as ***Type 2*** .
The name is temporary as further construction procedures are being developed by the author. A Type 1 factorization has been developed and introduced in an ongoing study by the author [Rosal, ongoing].

## V. Conclusion

The study of homogeneous factorization of a graph or digraph is a relatively new field. The construction procedure presented in this paper is part of the author's undergoing study on the development of construction procedures for homogeneous factorization of Hamming graphs.

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