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Some Fixed Point and Common Fixed Theorems in Banach Space for Rational Expressions

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Abstract:

In present paper we prove some fixed point and common fixed point theorems for non contractive mapping in rational expression in Banach space, which generalize the well known results.

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Key words: Banach space, Fixed point, Common fixed point, non contractive mapping

INTRODUCTION:

The study of non-contraction mapping concerning the existence of fixed points draws attention of various authors in non-linear analysis .It is well known that the differential and integral equations that arise in physical problems are generally non-linear, therefore the fixed point methods specially Banach contraction principle provides a powerful tool for obtaining the solutions of these equations which were very difficult to solve by any other methods.

It is well known that a Banach space is a linear space which is also in a special way a complete metric space. The combination of algebraic and metric structures opens up the possibility of studying linear transformation of one Banach space into another which has the additional property of being continuous . A normed linear space is a linear space N in which to each vector z , there corresponds a real number denoted by $\|x\|$ and called the norm of x in such a manner that

$(i)\|x\| \geq 0 and \|x\| = 0 \Leftrightarrow x = 0$

$(ii)\|x + y\| \leq \|x\| + \|y\|$

$(iii)\|ax\| = |a|\|x\|$

The non-negative real number $\|x\|$ is to be thought of as the length of vector x. If we regard $\|x\|$ as a real function defined on N. It is easy to verify that the normed function is called norm on N . It is easy to verify that the normed linear space N is a metric space w.r.to the metric d defined by $d(x, y) = \|x - y\|$.
A Banach space is a complete normed linear space.

1. DEFINITION

Banach Space :-

A Banach space is a vector space X over the field **R** of real numbers, or over the field **C** of complex numbers, which is equipped with a norm and which is complete with respect to that norm, that is to say, for every Cauchy sequence $\{x_n\}$ in X, there exists an element x in X such that

$$\lim_{n\to\infty} x_n = x$$

or equivalently:

$$\lim_{n\to\infty} \|x_n - x\|_X = 0$$

The vector space structure allows one to relate the behavior of Cauchy sequences to that of converging series of vectors. A normed space X is a Banach space if and only if each absolutely convergent series in X converges,

$$\sum_{n=1}^{\infty} \|v_n\|_X < \infty \quad \text{Implies that} \quad \sum_{n=1}^{\infty} v_n \quad \text{converges in X.}$$

Completeness of a normed space is preserved if the given norm is replaced by an equivalent one.
In this chapter, we prove the following theorems:-

2 MAIN RESULTS

Theorem 8.2.1

Let T be a continuous self mapping defined on a Banach space X , further T satisfies the following condition

$$\begin{aligned} \|Tx-Ty\| \le \alpha \frac{\|x-Tx\|\|y-Ty\|\|x-Ty\|+\|x-y\|^3}{\|x-y\|^2} \\ +\beta \frac{\|y-Ty\|\|y-Tx\|\|x-Ty\|+\|x-y\|^3}{\|x-y\|^2} \\ +\gamma\left[\|x-Tx\|+\|y-Ty\|\right] \\ +\delta\left[\|x-Ty\|+\|y-Tx\|\right]+\eta\|x-y\| \end{aligned} \qquad ....(8.2.1a)$$

$\forall x, y \in X$ *with* $x \neq y$ *with* $\|x-y\| \neq 0$ *and*
$\alpha+\beta+2\gamma+2\delta+\eta<1$ *where* $\alpha,\beta,\gamma,\delta,\eta \in [0,1[$.

Then T has unique fixed point.

Proof : Let x_0 be an arbitrary point in X and we define a sequence $\{x_n\}$ by means of iterates of T by setting $T^n(x_0)=x_1$ where n is a positive integer.

If $x_n = x_{n+1}$ for some n, then x_n is a fixed point of T. Taking for all n, then

$$\|x_{n+1}-x_n\| = \|Tx_n - Tx_{n-1}\|$$

$$\leq \alpha \frac{\|x_n - Tx_n\|\|x_{n-1}-Tx_{n-1}\|\|x_n - Tx_{n-1}\| + \|x_n - x_{n-1}\|^3}{\|x_n - x_{n-1}\|^2}$$

$$+\beta \frac{\|x_{n-1} - Tx_{n-1}\|\|x_{n-1}-Tx_n\|\|x_n - Tx_{n-1}\| + \|x_n - x_{n-1}\|^3}{\|x_n - x_{n-1}\|^2}$$

$$+\gamma\left[\|x_n - Tx_n\| + \|x_{n-1} - Tx_{n-1}\|\right]$$

$$+\delta\left[\|x_n - Tx_{n-1}\| + \|x_{n-1} - Tx_{n-1}\|\right]$$

$$+\eta\|x_n - Tx_n\|$$

$$= \alpha \frac{\|x_n - x_{n+1}\|\|x_{n+1}-x_n\|\|x_n - x_n\| + \|x_n - x_{n-1}\|^3}{\|x_n - x_{n-1}\|^2}$$

$$+\beta \frac{\|x_{n-1} - x_n\|\|x_{n-1}-x_{n+1}\|\|x_n - x_n\| + \|x_n - x_{n-1}\|^3}{\|x_n - x_{n-1}\|^2}$$

$$+\gamma\left[\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\|\right]$$

$$+\delta\left[\|x_n - x_n\| + \|x_{n-1} - x_{n+1}\|\right]$$

$$+\eta\|x_n - x_{n-1}\|$$

$$= [\alpha+\beta]\|x_n - x_{n-1}\| + \gamma\left[\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\|\right]$$

$$+\delta\left[\|x_{n-1} - x_n\| + \|x_n - x_{n+1}\|\right] + \eta\|x_n - x_{n-1}\|$$

$$= [\alpha+\beta+\gamma+\delta+\eta]\|x_{n-1} - x_n\| + \|x_n - x_{n+1}\|[\gamma+\delta]$$

So ,

$$\|x_{n+1}-x_n\| \le [\alpha+\beta+\gamma+\delta+\eta]\|x_{n-1}-x_n\| + [\lambda+\delta]\|x_n - x_{n+1}\|$$

$$\|x_{n+1}-x_n\| \le \frac{[\alpha+\beta+\gamma+\delta+\eta]}{1-\gamma-\delta}\|x_{n-1}-x_n\|$$

$$\|x_{n+1}-x_n\| \le s\|x_{n-1}-x_n\|$$

where

$$s = \frac{[\alpha+\beta+\gamma+\delta+\eta]}{1-\gamma-\delta} < 1, \qquad Since \;\; \alpha+\beta+2\gamma+2\delta+\eta<1$$

....

....

$$\|x_{n+1}-x_n\| \le s^n \|x_0 - x_1\|$$

By the triangle inequality , we have for m>n

$$\|x_n - x_m\| \le \|x_n - x_{n+1}\| + \|x_{n+1}-x_{n+2}\| + ... + \|x_{m-1}-x_m\|$$

$$\le \left[s^n + s^n + ... + s^n \right] \quad upto \;\; m-1$$

so

$$\|x_n - x_m\| \le \frac{\left[s^n\right]^{m-1}}{1-s}\|x_0 - Tx_0\|$$

$$\to 0 \;\; as \;\; m,n \to \infty$$

So $\{x_n\}$ is a Cauchy sequence in X so by the completeness of X, there is a point $v \in x$ such that $x_n \to v$ or $n \to \infty$.

But by continuity of T in X implies $Tv = T\left[\lim_{n\to\infty} x_n\right] = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = v$

V is fixed point of T in X.

Uniqueness: Suppose there is any other point w in X , where $w \ne v$ such that $T(w) = w$, then

$$\|v-w\| = \|Tv-Tw\|$$

$$\leq \alpha \frac{\|v-Tv\|\|w-Tw\|\|v-Tw\| + \|v-w\|^3}{\|v-w\|^2}$$

$$+\beta \frac{\|w-Tw\|\|w-Tv\|\|v-Tw\| + \|v-w\|^3}{\|v-w\|^2}$$

$$+\gamma\left[\|v-Tv\| + \|w-Tw\|\right]$$

$$+\delta\left[\|v-Tw\| + \|w-Tw\|\right]$$

$$+\eta\|v-w\|$$

$$= \left[2\delta+\eta\right]\|v-w\|$$

i.e

$$\|v-w\| < \left[2\delta+\eta\right]\|v-w\|$$

Which is a contradiction because $2\delta+\eta<1$.

So $v=w$. Hence fixed point is unique.

Theorem 8.2.2

Let T be self mapping defined on a Banach space such that (8.2.1a) holds. If for some positive integer p , T^p is continuous, then T has a unique fixed point.

Proof:

We define a sequence as in Theorem (8.2.1 a). Since it converges to some point v in X . Therefore its subsequence $\{x_{nk}\}$ where $(nk=np)$ also converges to v . Also

$$T^p(v) = T^p\left[\lim_{k\to\infty} x_{nk}\right] = \lim_{k\to\infty} T^p(x_{nk})$$

$$= \lim_{k\to\infty} x_{nk+1} = v$$

$$\therefore \quad T^p(v) = v$$

So v is fixed point of T^p .

Now we show that $Tv=v$. Let m be the smallest positive integer such that

$$T^m(v)=v$$

$$T^n(v)\neq v \quad for \quad n=1,2,3,.....,m-1$$

$$\|Tv-v\|=\left\|Tv-T\left(T^{m-1}v\right)\right\| \quad \text{because} \quad \mathrm{T}^m\mathrm{v}=\mathrm{v}$$

$$put \quad x=v, \; y=T^{m-1}v$$

$$\leq\alpha\frac{\|v-Tv\|\left\|T^{m-1}v-T\left(T^{m-1}v\right)\right\|\left\|v-T\left(T^{m-1}v\right)\right\|+\left[\left\|v-T^{m-1}v\right\|\right]^3}{\left[\left\|v-T^{m-1}v\right\|\right]^2}$$

$$+\beta\frac{\left\|T^{m-1}v-T\left(T^{m-1}v\right)\right\|\left\|T^{m-1}v-Tv\right\|\left\|v-T\left(T^{m-1}v\right)\right\|+\left[\left\|v-T^{m-1}v\right\|\right]^3}{\left[\left\|v-T^{m-1}v\right\|\right]^2}$$

$$+\gamma\left[\|v-Tv\|+\left\|T^{m-1}v-T\left(T^{m-1}v\right)\right\|\right]$$

$$+\delta\left[\left\|v-T\left(T^{m-1}v\right)\right\|+\right]\left\|T^{m-1}v-Tv\right\|$$

$$+\eta\left\|v-T^{m-1}v\right\|$$

$$=(\alpha+\beta+\gamma)\left\|T^{m-1}v-v\right\|+\delta\left\|T^{m-1}v-Tv\right\|$$

$$\leq(\alpha+\beta+\gamma)\left\|T^{m-1}v-v\right\|+\delta\left[\left\|T^{m-1}v-v\right\|+\|v-Tv\|\right]$$

$$=(\alpha+\beta+\gamma+\delta)\left\|T^{m-1}v-v\right\|$$

i.e

$$\|Tv-v\|<s_1\left\|T^{m-1}v-v\right\|$$

$$where \quad s_1=\alpha+\beta+\gamma+\delta$$

$$\because \; s_1<1$$

On continuous this process we get

$$\|Tv-v\|<(s_1)^m\|v-Tv\|$$

Which is a contradiction because $s_1<1$

So v is fixed point of T .

We can prove uniqueness as in Theorem 8.2.1

Now we further generalize the results of Theorem 8.2.1 in which T is neither continuous nor satisfies (8.2.1 a) In what condition T^m satisfies the same critical rational expression and continuous , where m is a positive integer , still T has unique fixed point.

Theorem 8.2.3.

Let T be a self map , defined on a Banach space X , such that for some positive integer m satisfies the conditions :

$$\begin{aligned}\left\|T^m(x)-T^m(y)\right\| \le\; & \alpha \frac{\left\|x-T^m x\right\|\left\|y-T^m y\right\|\left\|x-T^m y\right\|+\left[\left\|x-y\right\|\right]^3}{\left[\left\|x-y\right\|\right]^2} \\ & +\beta \frac{\left\|y-T^m y\right\|\left\|y-T^m x\right\|\left\|x-T^m y\right\|+\left[\left\|x-y\right\|\right]^3}{\left[\left\|x-y\right\|\right]^2} \\ & +\gamma\left[\left\|x-T^m x\right\|+\left\|y-T^m y\right\|\right] \\ & +\delta\left[\left\|x-T^m y\right\|+\left\|y-T^m x\right\|\right] \\ & +\eta\left\|x-y\right\|\end{aligned}$$

$\forall \quad x, y \in x, \quad x \neq y \quad$ with $\quad \left\|x-y\right\| \neq 0$

$\alpha, \beta, \gamma, \delta, \eta \in [0,1[\quad with\ \alpha+\beta+2\gamma+2\delta+\eta<1$

Then T has Unique fixed point in X .

Proof :

From the given condition of Theorem we assume that T^m has

$$T^m(v)=v$$

Unique fixed point v i.e again

$$\begin{aligned} T(v) &= T\left(T^m v\right) \\ &= T^{m+1}(v) \\ &= v \end{aligned}$$

We conclude that T is also a fixed point of T^m and T^m has unique fixed point v .

So v is unique fixed point of T .

Next we generalize Theorem (8.2.1) for three mapping F,G and T .

Theorem 8.2.4. : Let T and F be two self maps , defined on a Banach space X satisfying the conditions

$$\begin{aligned} \|T(x)-F(y)\| \le{}& \alpha \frac{\|x-Tx\|\|y-Fy\|\|x-Fy\|\|y-Tx\| + \left[\|x-y\|\right]^3}{\left[\|x-y\|\right]^2} \\ &+\beta \frac{\|y-Fy\|\|y-Tx\|\|x-Fy\| + \left[\|x-y\|\right]^3}{\left[\|x-y\|\right]^2} \\ &+\gamma\left[\|x-Tx\|+\|y-Fy\|\right] \\ &+\delta\left[\|x-Fy\|+\|y-Tx\|\right] \\ &+\eta\|x-y\| \qquad ....(8.2.4a) \end{aligned}$$

$\forall x, y \in X$, with x ≠ y with $\|x-y\| \neq 0$ *with* $\alpha,\beta,\gamma,\delta,\eta \in [0,1[$.

(8.2.4b) T and F are continuous on X .

(8.2.4c) There exist an $x_0 \in X$, such that in the sequence $\{x_n\}$, where

$$x_n = \begin{cases} Tx_{n-1} & \text{where } n \text{ is even} \\ Fx_{n-1} & \text{where } n \text{ is odd} \end{cases}$$

Then T and F have unique fixed point.

Proof : we have

$$\|x_{2n} - x_{2n+1}\| = \|Tx_{2n-1} - Fx_{2n}\|$$
$$\leq \alpha \frac{\|x_{2n-1} - Tx_{2n-1}\|\|x_{2n} - Fx_{2n}\|\|x_{2n-1} - Fx_{2n}\|\|x_{2n} - Tx_{2n-1}\| + \left[\|x_{2n-1} - x_{2n}\|\right]^3}{\left[\|x_{2n-1} - x_{2n}\|\right]^2}$$
$$+\beta \frac{\|x_{2n} - Fx_{2n}\|\|x_{2n} - Tx_{2n-1}\|\|x_{2n-1} - Fx_{2n}\| + \left[\|x_{2n-1} - x_{2n}\|\right]^3}{\left[\|x_{2n-1} - x_{2n}\|\right]^2}$$
$$+\gamma\left[\|x_{2n-1} - Fx_{2n}\| + \|x_{2n} - Fx_{2n}\|\right]$$
$$+\delta\left[\|x_{2n-1} - Fx_{2n}\| + \|x_{2n} - Tx_{2n-1}\|\right]$$
$$+\eta\|x_{2n-1} - x_{2n}\|$$

$$= \alpha \frac{\|x_{2n-1} - x_{2n}\|\|x_{2n} - x_{2n+1}\|\|x_{2n-1} - x_{2n+1}\|\|x_{2n} - x_{2n}\| + \left[\|x_{2n-1} - x_{2n}\|\right]^3}{\left[\|x_{2n-1} - x_{2n}\|\right]^2}$$
$$+\beta \frac{\|x_{2n} - x_{2n+1}\|\|x_{2n} - x_{2n}\|\|x_{2n-1} - x_{2n+1}\| + \left[\|x_{2n-1} - x_{2n}\|\right]^3}{\left[\|x_{2n-1} - x_{2n}\|\right]^2}$$
$$+\gamma\left[\|x_{2n-1} - Tx_{2n-1}\| + \|x_{2n} - Fx_{2n}\|\right]$$
$$+\delta\left[\|x_{2n-1} - Fx_{2n}\| + \|x_{2n} - Tx_{2n-1}\|\right]$$
$$+\eta\|x_{2n-1} - x_{2n}\|$$

$$\|x_{2n} - x_{2n+1}\| \leq \frac{(\alpha + \beta + \gamma + \delta + \eta)}{1 - \delta - \gamma}\|x_{2n-1} - x_{2n}\|$$
$$\|x_{2n} - x_{2n+1}\| \leq l\,\|x_{2n-1} - x_{2n}\|$$
where
$$l = \frac{(\alpha + \beta + \gamma + \delta + \eta)}{1 - \delta - \gamma} < 1$$

....

....

$$\|x_{2n} - x_{2n+1}\| \leq l^{2n}\|x_0 - x_1\|$$
$$\|x_{2n+1} - x_{2n+2}\| \leq l^{2n+1}\|x_0 - x_1\|$$

So $\{x_n\}$ is a Cauchy sequence.

By the completeness of X, $\{x_n\}$ converges to a point X. Suppose $\{x_n\} \to p$,

And then the subsequence $\{x_{nk}\}$ also converges to P .

Now

$$TF(P) = TF\left(\lim_{k\to\infty} x_{nk}\right) = \lim_{k\to\infty} x_{nk+1} = P$$

Now we will prove that

$$F(P) = P$$

Suppose $F(P) \neq P$ then

$$\|P - F(P)\| = \|TF(P) - F(P)\|$$

$$\leq \alpha \frac{\|F(P) - TF(P)\| \|P - F(P)\| \|F(P) - F(P)\| \|P - TF(P)\| + \left[\|F(P) - P\|\right]^3}{\left[\|F(P) - P\|\right]^2}$$

$$+\beta \frac{\|P - F(P)\| \|P - TF(P)\| \|F(P) - F(P)\| + \left[\|F(P) - P\|\right]^3}{\left[\|F(P) - P\|\right]^2}$$

$$+\gamma\left[\|F(P) - F(P)\| + \|P - F(P)\|\right]$$

$$+\delta\left[\|F(P) - F(P)\| + \|P - TF(P)\|\right]$$

$$+\eta\|F(P) - P\|$$

$$= (\alpha + \beta + \eta)\|F(P) - P\| + \gamma\left[\|F(P) - TF(P)\| + \|P - F(P)\|\right]$$
$$+\delta\|P - TF(P)\|$$

$$\leq (\alpha + \beta + \eta)\|F(P) - P\| + \gamma\left[\|F(P) - P\| + \|TF(P) - P\| + \|P - F(P)\|\right]$$
$$+\delta\|P - TF(P)\|$$

$$= (\alpha + \beta + \eta + 2\gamma)\|P - F(P)\|.$$

But $\alpha + \beta + \eta + 2\gamma < 1.$

So

$$\|F(P) - P\| < \|P - F(P)\|$$

Which is a contradiction. So

$F(P)=P$

Now $TF(P)=T\left[F(P)\right]=T(P)=P$

That is a common fixed point of T and F .

Uniqueness :

If possible let q (where $q\neq p$) be another common fixed point of F and T .

i.e .

$$F(q)=T(q)=q$$

Now

$$\|P-q\|=\|TP-Fq\|$$

$$\leq\alpha\frac{\|P-TP\|\|q-FP\|\|P-Fq\|\|q-TP\|+\left[\|P-q\|\right]^3}{\left[\|P-q\|\right]^2}$$

$$+\beta\frac{\|q-FP\|\|q-TP\|\|P-Fq\|+\left[\|P-q\|\right]^3}{\left[\|P-q\|\right]^2}$$

$$+\gamma\left[\|P-TP\|+\|q-Fq\|\right]$$

$$+\delta\left[\|P-F(q)\|+\|q-TP\|\right]$$

$$+\eta\|P-q\|$$

$$=\left(\alpha+\beta+2\delta+\eta\right)\|P-q\|$$

so

$p=q,\quad \sin ce\ \alpha+\beta+2\delta+\eta<1$

Hence fixed point is unique.

Theorem 8.2.5 Let F,G and T be three self mappings of a Banach space X , satisfies

$$\|GF(x)-TF(y)\| \le \alpha \frac{\|x-GF(x)\|\|y-TF(y)\|\|x-TF(y)\|\|y-GF(x)\|+\left[\|x-y\|\right]^3}{\left[\|x-y\|\right]^2}$$
$$+\beta \frac{\|y-TF(y)\|\|y-GF(x)\|\|x-TF(y)\|+\left[\|x-y\|\right]^3}{\left[\|x-y\|\right]^2}$$
$$+\gamma\left[\|x-GF(x)\|+\|y-TF(y)\|\right]$$
$$+\delta\left[\|x-TF(y)\|+\|y-GF(x)\|\right]$$
$$+\eta\|x-y\|$$

If $\alpha+\beta+3\gamma+2\delta+\eta<1.$

So G, F, T has *common fixed po*int .

Proof : Let x_0 be an arbitrary point and we defined a sequence

$$x_{2n+1}=GF(x_{2n}),\ x_{2n+2}=TF(x_{2n+1})$$

Now

$$\|x_{2n+1}-x_{2n+2}\|=\|GF(x_{2n})-TF(x_{2n+1})\|$$

$$\le \alpha \frac{\|x_{2n}-GF(x_{2n})\|\|x_{2n+1}-TF(x_{2n+1})\|\|x_{2n}-TF(x_{2n+1})\|\|x_{2n+1}-GF(x_{2n})\|+\left[\|x_{2n}-x_{2n+1}\|\right]^3}{\left[\|x_{2n}-x_{2n+1}\|\right]^3}$$

$$+\beta \frac{\|x_{2n+1}-TF(x_{2n+1})\|\|x_{2n+1}-GF(x_{2n})\|\|x_{2n}-TF(x_{2n+1})\|+\left[\|x_{2n}-x_{2n+1}\|\right]^3}{\left[\|x_{2n}-x_{2n+1}\|\right]^2}$$

$$+\gamma\left[\|x_{2n}-GF(x_{2n+1})\|+\|x_{2n+1}-TF(x_{2n+1})\|\right]$$

$$+\delta\left[\|x_{2n}-TF(x_{2n+1})\|+\|x_{2n+1}-GF(x_{2n})\|\right]$$

$$+\eta\|x_{2n}-x_{2n+1}\|$$

$$= \alpha \frac{\|x_{2n} - x_{2n+1}\| \|x_{2n+1} - x_{2n+2}\| \|x_{2n} - x_{2n+2}\| \|x_{2n+1} - x_{2n+1}\| + \left[\|x_{2n} - x_{2n+1}\|\right]^3}{\left[\|x_{2n} - x_{2n+1}\|\right]^2}$$

$$+\beta \frac{\|x_{2n+1} - x_{2n+2}\| \|x_{2n+1} - x_{2n+1}\| \|x_{2n} - x_{2n+2}\| + \left[\|x_{2n} - x_{2n+1}\|\right]^3}{\left[\|x_{2n} - x_{2n+1}\|\right]^2}$$

$$\begin{aligned}
&+\gamma\left[\|x_{2n} - x_{2n+2}\| + \|x_{2n+1} - x_{2n+2}\|\right] \\
&+\delta\left[\|x_{2n} - x_{2n+2}\| + \|x_{2n+1} - x_{2n+1}\|\right] \\
&+\eta\|x_{2n} - x_{2n+2}\| \\
&\leq \alpha\|x_{2n} - x_{2n+1}\| + \beta\|x_{2n} - x_{2n+1}\| \\
&+\gamma\left[\|x_{2n} - x_{2n+1}\| + \|x_{2n+1} - x_{2n+2}\| + \|x_{2n+1} - x_{2n+2}\|\right] \\
&+\delta\left[\|x_{2n} - x_{2n+1}\| + \|x_{2n+1} - x_{2n+2}\|\right] + \eta\|x_{2n} - x_{2n+1}\|
\end{aligned}$$

$$= (\alpha + \beta + \gamma + \delta + \eta)\|x_{2n} - x_{2n+1}\| + (2\gamma + \delta)\|x_{2n+1} - x_{2n+2}\|$$

$$\|x_{2n+1} - x_{2n+2}\| \leq \left(\frac{\alpha + \beta + \gamma + \delta + \eta}{1 - 2\gamma - \delta}\right)\|x_{2n} - x_{2n+1}\|$$

$$\|x_{2n+1} - x_{2n+2}\| \leq l\|x_{2n} - x_{2n+1}\|$$

$$\text{where} \quad l = \left(\frac{\alpha + \beta + \gamma + \delta + \eta}{1 - 2\gamma - \delta}\right) < 1$$

$$\text{because} \quad \alpha + \beta + 3\gamma + 2\delta + \eta < 1$$

Proceeding in the same manner, we get

$$\|x_{2n+1} - x_{2n+2}\| \leq l\|x_{2n} - x_{2n+1}\| \leq l^2\|x_{2n-1} - x_{2n}\| \ldots\ldots\ldots \leq l^{2n+1}\|x_0 - x_1\|$$

So $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, there is a point $p \in X$ such that $x_n \to p$ as $n \to \infty$. Now we assume that

$$p \neq TF(p), \quad then \quad \|p - TF(p)\| > 0$$

$$\|P - TF(P)\| \leq \|P - x_{2n+1}\| + \|x_{2n+1} - TF(P)\|$$

$$= \|GF(x_{2n}) - TF(P)\| + \|P - x_{2n+1}\|$$

$$\leq \alpha \frac{\|x_{2n} - GF(x_{2n})\| \|P - TF(P)\| \|x_{2n} - TF(P)\| \|P - GF(x_{2n})\| + [\|x_{2n} - P\|]^3}{[\|x_{2n} - P\|]^2}$$

$$+\beta \frac{\|P - TF(P)\| \|P - GF(x_{2n})\| \|x_{2n} - TF(P)\| + [\|x_{2n} - P\|]^3}{[\|x_{2n} - P\|]^2}$$

$$+\gamma[\|x_{2n} - GF(x_{2n})\| + \|P - TF(P)\|]$$
$$+\delta[\|x_{2n} - TF(P)\| + \|P - GF(x_{2n})\|]$$
$$+\eta[\|x_{2n} - P\| + \|P - x_{2n+1}\|]$$

$$= \alpha \frac{\|x_{2n} - x_{2n+1}\| \|P - TF(P)\| \|x_{2n} - TF(P)\| \|P - x_{2n+1}\| + [\|x_{2n} - P\|]^3}{[\|x_{2n} - P\|]^2}$$

$$+\beta \frac{\|P - TF(P)\| \|P - x_{2n+1}\| \|x_{2n} - TF(P)\| + [\|x_{2n} - P\|]^3}{[\|x_{2n} - P\|]^2}$$

$$+\gamma[\|x_{2n} - x_{2n+1}\| + \|P - TF(P)\|]$$
$$+\delta[\|x_{2n} - TF(P)\| + \|P - x_{2n+1}\|]$$
$$+\eta[\|x_{2n} - P\| + \|P - x_{2n+1}\|]$$

$$\|P - TF(P)\| \leq \gamma \|P - TF(P)\|$$

Which is a contradiction because $\alpha+\beta+3\gamma+2\delta+\eta<1$.

So $TF\left(P\right)=P$.

Similarly we assume $P\neq GF$ we get a contradiction.

Hence $TF\left(P\right)=GF\left(P\right)=P$

So P is common fixed point of TF *and* GF .

So P is common fixed point of $G,\ F$ *and* T as we proved before Uniqueness can be proved easily .