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Symmetric Spaces as Riemannian Manifolds

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Abstract:

This paper tackles some geometrical and algebraic approaches, particularly lie algebra, disclosing some of their algebraic and topological properties with some concentration on lie algebras and root systems. The main aims of this study are to explain some algebraic features of symmetric spaces and how to get some of their properties using algebraic approach. The paper, accordingly, reflects some conclusions on the foundation of symmetric spaces with logical ordering of notions and consequences that symmetric space is crucial field for understanding abstract and applied features of spaces for further future studies.

Key words: Topological Manifolds, Riemannian Manifold, Lie groups, Lie algebras, Root systems, Homogeneous space, Curvature Tensor, Symmetric spaces.

1. INTRODUCTION:

In studying spaces, one of the aims of this study is to introduce spaces that can suit some scientific applications. Many scientific problems in various fields may have their own conditions that might not agree with the geometric structure and properties of some spaces familiar to mathematicians and geometers.

In differential geometry, manifolds do their role to meet some scientific demands. This is due to some properties of manifolds.

Differentiable manifolds for example lead to Lie groups, and these with their Lie algebras help to introduce some abstract spaces such as symmetric spaces. Many properties of symmetric spaces can be studied through their Lie algebras and root systems.

In this study we are going to discuss some of these approaches which lead to some invariant properties of symmetric spaces. We can look to these spaces from different aspects which unify some algebraic and geometric.

2. TOPOLOGICAL MANIFOLD:

Manifolds are topological spaces that are locally Euclidean and Hausdorff . A smooth manifold M is a manifold endowed with a smooth structure, that is an atlas of charts satisfying smoothness conditions**.**

2.1 Definition: A manifold M of dimension n, or n-manifold is topological space with the following properties :

(I) M is Housdorff space.

(II) M is locally Euclidean of dimension n and,

(III) M has a countable basis of open sets .

As a matter of notion dim M is used for the dimension of M, when $\dim M = 0$, then M is accountable space with the discrete topology.

2.2Examples:

Simple examples of smooth manifolds are the Euclidean space n- space R^n and unit sphere S^n .

dim $M = 0$, then M is a countable space with discrete topology .

***Theorem*[6]:**
A topological manifold $\boldsymbol{M}$ is locally connected, locally compact and union of a countable collection of compact subsets further more it is normal and materializable .

3. RIEMANNIAN MANIFOLD:

In this section we introduce the notion of a Riemannian manifold $(M.g)$.

Let M be a smooth manifold with a metric g , the pair (M,g) is called a Riemannian manifold . Geometric properties of (M,g) which only depend on the metric$g$ are called intrinisic or metric properties.

3.1 Definition:

(a) A Riemannian manifold is a pair $(M.g)$ consisting of a smooth manifold M and a metric g on the tangent bundle ,i.e., a smooth , symmetric positive definite (0,2) -tensor field on M. The tensor g is called a Riemannian metric on M .

(b) Two Riemannian manifolds $M_i, M_i)$ $(i = 1,2)$ are said to be isometric if there exists a diffeomorphism $\Phi : M_1 \rightarrow M_2$ such that $\Phi^* g_2 = g_1$.

3.2 Example (The Euclidean space) :

The space R^n has a natural metric

$$g_0 = (dx^1)^2 + \cdots + (dx^n)^2.$$

The geometry of (R^n, g_0) is the classical Euclidean geometry.

4. LIE GROUPS AND LIE ALGEBRAS:

Lie groups and their Lie algebras are very useful and important tools when studying symmetric spaces , this results from the fact that their algebraic properties derive from the group axioms , and their geometric properties derive from the identification of group operation s with points in a topological spaces , and these are manifolds . We should note that every Lie group is a smooth manifold **.**

4.1 Definition (Lie group):

Alie group G is a group satisfying two additional axioms:

(i) The mapping of the group operation $G \times G \to G$ defined by
$(x , y) \to xy$ and

(ii) The invers map $G \to G$ defined by $x \to x^{-1}$ are both smooth

4.2 Examples:

(1) The general linear group $Gl(n, \mathbb{R})$ is the set of invariable $n \times n$ matrix with real entries. It's a group under matrix multiplication, and it's an open supmanifold of the vector space $M(n, \mathbb{R})$.

(2) The real number field $\mathbb{R}$ and Euclidean space $\mathbb{R}^n$ are Lie group under addition , because the coordinates of x, y are smooth (linear) functions of (x, y) .

4.3 Lie algebra:

In this section we review basic concepts of Lie algebras besides some of their properties in studying symmetric spaces.

4.4 Definition:

A Lie algebra is vector space V over a field F where multiplication is the lie bracket $[x, y]$ and the following properties are satisfied :

(1) if $x, y \in V$ then $[x, y] \in V$.

(1) $[x, y] = -[y, x]$

(2) $[x, \alpha y + \beta z] = \alpha[x, y] + \beta[x, z] \; for \; \alpha, \beta \in F$

(3)$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (The Jacobi identity)

Every Lie group G has a corresponding Lie algebra denoted by $\mathbf{g}$, it is the tangent space at the identity of the Lie group G . The Lie algebra generates a group through the exponential mapping.

3.5 Example:
The Lie algebra g of R^n as a Lie group is again R^n,where $[x, y] = 0\ \forall x, y \in g$ .Thus the Lie bracket for the Lie algebra of any abelian group is $zero$.

4. ROOT SYSTEMS:

The root systems are very effective tools which are used in classifying and studying the structure of Lie algebras .

4.1 Definition (Root systems):
Let v be a real finite - dimensional vectors space and $R \subset V$ a finite set of nonzero vectors ,R is called a root systems in V (and its members called roots) if
(i) R generates V.
(ii) For each $\propto \in R$ there exists a reflection $S_\propto$ along $\propto$ leaving Rinvariant.
(iii) For all $\propto$, β ∈ R the number $a_{\mathcal{B},\propto}$ determined by
$s_\propto \beta = \beta - a_{\mathcal{B},\propto} \propto$ is an integer that is $a_{\mathcal{B},\propto} \in$ z .

4.2: *Theorem*[3]
(i) Each root system has a basis
(ii) Any tow bases are conjugate under a unique weyl group element
(iii) $a_{B,\propto} \leq 0$ for any tow different element $\propto, \mathcal{B}^/$, in the same basis

5. HOMOGENEOUS SPACE:

A manifold M is said to be a **homogeneous space** of the Lie group G if there is a transitive C^{∞} action of G on M .

5.1 Definition (**Homogeneous space**):
Homogeneous space s is a transitive $G-space$ that is isomorphic to a quotient space $G/_H$That is there is an isomorphism m by the mapping Φ as in the previous theorem making $G/_H$ isomorphic to X. In a homogeneous space every point looks exactly like every other point .We can also look at a homogeneous space as space whose isometry group acts transitively on it .

5.2 Example :
Let G be a topological group and H a normal subgroup of G . Then G is topological transformation group on G/H , and G acts transitively on G/H .

5.3 Homogeneous Riemannian Space:

5.4 Definition:
Ahomogeneous Riemannian space is a Riemannian manifold (M,g) whose isometry group Isom(M,g) acts transitively on M.

6. MANIFOLD OF CONSTANT CURVATURE:

The manifold of constant curvature, it's a simplest Riemannian manifold.

6.1 Definition:
We recall that the Riemannian manifold M is said to have constant curvature if all sectional curvature at all points have the same constant value K. We suppose M to be a Riemannian manifold and let ω^i ,$1 \leq i \leq n$, denote the field of co-frames dual

to an orthogonal frame field $E_1, \dots, E_n$ on an open set $U \subset M$,with ω_i^j ,$1 \leq i, j \leq n$, denoting the corresponding connection forms.

7. SYMMETRIC S PACES

These are spaces which possess the properties of symmetry and homogeneousness, and they have many applications, this is because they have mixed algebraic and geometric properties. The beginning for these spaces is that they are spaces with parallel curvature tensor, later they were introduced through different approaches. They have much in common. Any symmetric space has its own special geometry, Euclidean, elliptic and hyperbolic are some of these geometries. We give some approaches to symmetric spaces using some algebraic and geometric properties.

7.1Some Approaches to Symmetric S paces:

Asymmetric space can be considered as:

A Riemannian manifold with point reflection, a Riemannian manifold with parallel curvature tensor, a Lie group with certain involution, a homogeneous space with special isotropy group, a Riemannian manifold with special holonomy, a special Killing vector field , a Lie triple system.

These may be some of many other approaches to symmetric spaces, but we are interested in this work in some of these approaches to reach the required aims of this study .

7.2 Definition (Locally Symmetric Space):

A Riemannian manifold $\boldsymbol{M}$ is called locally symmetric if its curvature tensor is parallel

7.3 Definition:

A Riemannian manifold M is called a Riemannian locally symmetric space if for each $P \in M$ there exists a normal neighbourhood of P on which the geodesic symmetry with respect to P is an isometry.

7.4 $\boldsymbol{Theorem}$[1] :
M is locally symmetric if and only if there exists a symmetric space S such that M is locally isometric to S.

8. MAIN RESULTS:

1- The elements of a Lie group can act as transformations on the elements of the symmetric space.
2- Every lie algebra corresponds to a given root system and each symmetric space corresponds to a restricted root system
3- The geometric and algebraic approaches to symmetric spaces can be modified to deduce each other.
4- Most of features of symmetric spaces can be extracted from lie algebras.
5- The study of symmetric spaces and continuous research in their properties and classification can lead to most surprising results that can help in their applications

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