EUROPEAN ACADEMIC RESEARCH
Vol. VI, Issue 2/ May 2018

Impact Factor: 3.4546 (UIF)
DRJI Value: 5.9 (B+)

ISSN 2286-4822
www.euacademic.org

Symmetric Spaces as Lie Groups and Lie Algebras

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Abstract:

In this paper we have introduced Lie groups and Lie algebras which help to give further understanding of symmetric spaces and help scientists who are seeking for suitable spaces for their applications. One of the aims of this study is to put forward the close connection between different approaches to symmetric spaces namely algebraic and geometrical features of these spaces with some results. We deal with the basic concept of a root system. First, its origins in the theory of Lie algebras are exposed, then an axiomatic definition is provided. So this Paper is an attempt to disclose some of these features and helps in more understanding and put forward a base for future applications.

Key words: Manifolds, Tensor Field, Lie groups, Lie algebras, Root systems, Cartan Matrix, Dynkin diagrams, Symmetric spaces.

1. INTRODUCTION:

Nevertheless when introducing symmetric spaces one can not ignore the fact that a symmetric space M can be introduced as a homogeneous space $G/_H$ where G is its group of isometries, which is a Lie group, and H is the isotropy subgroup. Many properties of symmetric spaces can be studied through their Lie algebras and root systems, and specially the problem of classification of symmetric spaces. The goal of this review paper is to disclose some relations and results related to root systems, Cartan matrices, Dynkin diagrams and finally discussing the problem of classification of Lie algebras. Many authors contributed their efforts to developing this issue, and introduced various applications of Lie algebras and symmetric spaces in different fields especially in physics.

In mathematical context in this paper we are treating some algebraic and topological properties of Lie algebras associated to symmetric spaces to make it possible for further understanding and carrying more applications.

2. THE MANIFOLDS:

Manifolds $M$are topological spaces that are locally Euclidean and Hausdorff. A smooth manifold is a manifold endowed with a smooth structure, which is an atlas of charts satisfying smoothness conditions.

2.1 Definition:

A manifold M of dimension n , or n-manifold is topological space with the following properties.

(i) M is Housdorff space.

(ii) M is locally Euclidean of dimension n and,

(iii) M has a countable basis of open sets .

As a matter of notion dim M is used for the dimension of M , when

dim M = 0 , then M is a countable space with discrete topology.

2.2 Examples:

1. (Circle) . Define the circle $S^1 = \{z \in \mathbb{C}: |z| = 1\}$. Then for any fixed point $z \in S^1$,write it as $z = e^{2\pi ic}$ for a unique real number $0 \leq c \leq 1$, and define the map

$$v_z: t \to e^{2\pi it} .$$

We note that v_z maps the natural $\mathrm{I_c} = (\mathrm{c} - \frac{1}{2}, \mathrm{c} + \frac{1}{2})$ to the neighborhood of z given by $\mathrm{s}^1/-\mathrm{z}$, and it is a homeomorphism. Then $\varphi_z = v_z|_{\mathrm{I_c}}^{-1}$ is a local coordinate chart near.

By taking products of coordinate charts , we obtain charts for the Cartesian product of manifolds. Hence the Cartesian product is a manifold .

2.3 Theorem[5] :

A topological manifold M is locally connected, locally compact, and a union of a countable collection of compact subsets; furthermore, it is normal and metrizable.

3. Tensor Field:

Once the covariant derivative is defined for fields of vectors it can be defined for arbitrary tensor fields using the following identities where φ and ψ are any two tensors :

$$\nabla_v(\varphi \otimes \psi) = (\nabla_v \varphi) \otimes \psi + \varphi \otimes (\nabla_v \psi) ,$$

and if φ and ψ are tensor fields of the same tensor bundle then

$$\nabla_v(\varphi + \psi) = \nabla_v \varphi + \nabla_v \psi .$$

3.1 Examples:

For scalar fields $\emptyset$, covariant differentiation is simply partial differentiation:

$$\emptyset_j a = \partial_a \emptyset .$$

For a covariant vector field λ_a , we have :

$$\lambda_{ac} = \partial_{\mathrm{c}} \lambda_a - \Gamma^{\mathrm{b}}_{\mathrm{ac}} \lambda_b .$$

For a type (2,0) tensor field T^{ab} , we have :

$$T^{ab}_c = \partial_c T^a_b + \Gamma^a_{dc} T^{db} + \Gamma^b_{dc} T^{ad} .$$

For a type (1,1) tensor fields T^a_b , we have :

$$T^{ab}_c = \partial_c T^a_b + \Gamma^a_{dc} T^d_b - \Gamma^d_{bc} T^a_d .$$

The notion above is mean in the sense

$$T^{ab}_c = (\nabla_{ec} T)^{ab}$$

4. Lie Groups

Lie groups and their Lie algebras are very useful and important tools when studying symmetric spaces , this results from the fact that their algebraic properties derive from the group axioms , and their geometric properties derive from the identification of group operation s with points in a topological spaces , and these are manifolds. We should note that every Lie group is a smooth manifold**.**

4.1 Definition:

A Lie group G is a group satisfying the well known axioms of group ,besides the mappings $G \times G \longrightarrow G \quad and \quad G \longrightarrow G^{-1}$ defined by $(x, y) \longrightarrow xy \;\; and \;\; x \longrightarrow x^{-1}$ respectively are both C^{∞} mappings .This definition implies that the Lie group G is a differentiable manifold . Lie groups are very important due to the fact that , their algebraic properties derive from group axioms , and their geometric properties derive from the identification of group operations with points in a topological space .

4.2 Example

The set $Gl(n, R)$ of nonsingular $n \times n$ matrices is a group with respect to matrix multiplication. An $n \times n$ matrix X is nonsingular if and only if $\det X \neq 0$

If $X, YGl \in (n, R)$ then both the maps $(x, y) \longrightarrow xy \quad and$ $x \longrightarrow x^{-1}$ are C^{∞}. Thus $Gl(n, R)$ is a Lie group .

4. 3 Lemma[3]:

Suppose G is a smooth manifold with a group structure such that the map

$G \times G \rightarrow G$ given by $(g\,, h) \rightarrow gh^{-1}$ is smooth . Then G is a lie group .

5. The Lie algebra

In this section , we review basic concepts of Lie algebras , besides some of their properties needed in studying symmetric spaces.

5.1 Definition :

A Lie algebra is a pair $(V, [\,,])$ where V is a vector space ,and $[\ ,\ ]$ is a Lie bracket , $[\ ,\ ]: V \times V \to V$ satisfying :

(1) $[v, w] = -[w, v]$ skew-symmetric .

(2) $[av + bu, w] = a[v, w] + b[u, w]$ a bilinear .

(3) $[v, [w, u]] + [w, [u, v]] + [u, [v, w]] = 0$

For all $, u \; and \; w \in V$. a Bianchi identity .

A Lie Bracket is a binary operation $[\ ,\ ]$ on a vector space V

5.2 Example :

Let $V = R^3$, $[\ ,\ ]: R^3 \times R^3 \to R^3$ as proved that it is a Lie algebra.

5.3 Example:

A homeomorphism of Lie algebra ℓ is a linear map $\varphi: \ell \longrightarrow \ell$ preserving the Lie bracket. This means that $\varphi[\ell_1, \ell_2] = [\varphi(\ell_1)\, \varphi, (\ell_2)]$ for any $\varphi(\ell_1, \ell_2) \in \ell_1 \times \ell_2$

proof:

$$\varphi[\ell_1, \ell_2] = \varphi[(\ell_1 \ell_2 - \ell_2 \ell_1)] = [\varphi(\ell_1)\ \varphi(\ell_2) - \varphi(\ell_2)\ \varphi(\ell_1)]$$
$$= [\varphi(\ell_1)\, \varphi, (\ell_2)]$$

which shows the claimed linearity and preserving the bracket in ℓ . It worth mentioning that the homomorphism in this example is the same as homomorphism defined between groups as general, and as we know, the Lie algebra of a Lie group can be seen as the vector space at the identity element of its Lie group.

5.4 Theorem[3]:

Let G be a Lie group and ℓ its Lie algebra :

(1) If H is a Lie subgroup of G , η is a Lie subalgebra of ℓ .

(2) If η is a Lie subalgebra , there exists a unique Lie subgroup H of G such that Lie algebra of H is isomorphic to η.

5.5 some Properties of a Lie algebra:

Lie algebras have many properties related to their effects on linear operators. As we know a Lie algebra is a linearization of its Lie group, that is, a Lie algebra is a linear vector space in which linear operations can be carried on it easily, rather than on its original Lie algebra. So we give some properties of Lie algebras related to linear operators below

(i) The operators in a Lie algebra form a linear vector space.

(ii) The operators closed under commutation: the commutator of two operators is in the Liealgebra;

(iii) The operators satisfy the Jacobi identity.

6. The Lie Algebra of a Lie Group:

6.1 Definition:

A lie group is a group G which is also on analytic manifold such that mapping

$(\partial, \tau) \rightarrow \partial r^{-1}$ of the product manifold

$G \rightarrow G$ in to G is analytic.

6.2 Definition (Lie Algebra of a Lie Group):

The tangent space to a linear Lie group G at the identity$e$ denoted $g = T_e G$ is its Lie algebra endowed with a (non-associative) a multiplication the Lie bracket satisfying the axioms of a Lie algebra as a vector space.

6.3 Definition (Lie algebra morphism/isomorphism):

i. Let $\phi : G_1 \rightarrow G$be amapping between two groups .Then ϕ is a group homomorphism if $\phi(ab) = \phi(a)\phi(b)$for all $a, b \in G_1$. If ϕ it is also bijective it is called a group isomorphism

ii. Let $\phi\ : G_1\ \to G_2$ a mapping between two lie groups .If ϕ is smooth(or analytic in the case of complex lie group)and a group homomorphism it is alie group homomorphism .If ϕ is also diffeomorphic (bijective and $\phi\,,\phi^{-1}$ both smooth) it is called alie group isomorphism .

iii. Let $\phi\ : G_1\ \to G_2$ a mapping between two Lie algebras.

If ϕ is linear and it preserves the lie bracket namely $[\ \phi(a), \phi\ (b)] = \phi[a, b]$ for all $a, b \in G_1$it is called alie algebras morphism . it is also bijective it is a Lie algebras isomorphism.

6.4 Theorem[4] (connected Lie group of a given Lie algebra):

Let g be afint dimensional lie algebra .Then there a unique connected and simply conneted lie group $G\ with\ g$ as it is lie algebra .If $G^{/}$ is another connected lie groupwith this lie algebra it is of the form $^{G}/_{\mathcal{Z}}$ wher $\mathcal{Z}$ is some discrete central subgroup of G **.**

6.5 Example:

let G be of all isometries of R^2. if S is the symmetry of R^2 with respect to aline then $G = G^{U}G_S(disjoint\ union)$ we can turn G_S into an analytic by requiring the mapping $\partial \to S_\partial(\ \partial \in\ G)$ to be an analytic diffeormorphism of G onto G_S . This makes G a lie group . On the other hand if G_1 and are two components of a lie group G and $x_1 G_1$, $x_2 \in G_2$ then $g\!\to\ x_2\, x_1^{-1} g$ ananalytic diffeormorphism of G_1 on to G_2 .

6.6 Definition:

A one – parameter subgroup of a Lie a group G is an analytic homomorphism of R in to G.

6.7 Theorem[4] :

let G be aliegroup with liealgebra g . The exponential mapping of the manifold g in to G has the differential $d\exp_x = d(lex_p\ x)$, o $\frac{1-e^{-adx}}{adx}(x \in g)$. As usual, g is here identified with tangent space g_x.

6.8 Definition (Lie Subalgebra and the Ideal) :

If g is a Lie algebra , a subalgebra η of g is a subspace o g such that $[u, v] \in$ η $\forall\, u, v \in$ η . If η is a subspace of g such that that $[u, v] \in$ η $\forall\, u \in$ η and all , $v \in$ η we call η an ideal in g . Note that many properties of Lie groups structure can be studied and derived through their Lie algebras , that is why they are important to be studied . A simple Lie algebra has no proper ideal. The semisimple algebras are constructed of simple ones.

7. Killing Form

If g is a Lie algebra , we define the Killing form B of g over afield F as the bilinear form $B: g \times g \longrightarrow F$, $(X, Y) \longrightarrow tr(ad\, X \circ ad\, Y)$. The Lie group and its Lie algebra are called semisimple if the Killing form is non degenerate .

7.1 Definition(killing form):

The killing form is said to be non degenerate if : $\forall\ y = 0$, $k(x\,, y) = 0$, Implies $x = 0$

7.2 Definition(representation):

A representation of a Lie algebra is a lie algebra homomorphism from g to the Lie algebra $EN\,d$(v): p : $g \rightarrow$ gt (v).

7.3 Definition(a map):

For a lie algebra g and any $x \in g$ we define a map

$$ad_x : g \to g \,,$$

$y \longleftarrow [x, y]$ which is the a djoint action .Every lie algebra has representation on itself the a djoint representation defined via the map ad : $g \to g$t (g) ,$x \to ad_x$.

7.4 Theorem[4] (Cartan criterion):
A Lie algebra over a field F of characteristic zero is semi simple if and only if the killing from is non degenerate .

7.5 Theorem[4] (Cartan's first criterion):
A Lie algebra g is solvable if and only if $k(x\,, y) = 0$,

$$for\ all\ x \in [g, g], y \in g\ .$$

7.6 Theorem[4] (Cartan's second criterion):
A Lie algebra $\boldsymbol{g}$ is semi simple if and only if its killing form is non degenerate .

8. Simple and semi simple Lie algebras:

8.1 Definition (Simple and semi simple Lie algebras):
A simple Lie algebra g has no proper ideals or in other words a simple Lie algebra has no ideals except itself and **0** and$[g, g] \neq 0$. A semi simple Lie algebra is the direct sum of simple algebras , and has no proper abelian ideal If g is simple then Z(g) = 0 and $[g, g] = g$.

When a Li algebra g inot simple we can factor out anon zero proper ideal h to get a Lie algebra of smaller dimension which we call it a quotient algebra denoted by g/H .

9. Derived algebra:

9.1 Definition(Derived algebra):
It is the collection of all linear combinations of $[x, y\,]\ \forall\, x, y \in g$ and it is denoted by $[g, g]$. It is also an ideal and

determines whether the Lie algebra is a belian or not , in fact we can say that the Lie algebra g is a belian if its derived algebra is the zero vector.

9.2 Definition (The Radical):

In the Lie algebra g the unique maximal solvable ideal is called the radical of gdenoted rad g . Suppose g is an arbitrary Lie algebra , r is an ideal included in no larger solvable ideal and hany other solvable ideal of g. we have $r + h = r$ which means $h \subset r$ and r is unique. It can be shown that a Lie algebra gis semi simple if rad $g = 0$ and a simple algebra g is also semisimple but the converse is not true.

9.3 Proposition[5](Radicals):

Every Lie algebra g contains a unique largest solvable ideal. This ideal rad (g) is called the radical of g .

10 Solvable Lie algebra:

10.1 Definition(Solvable Lie algebra):

A lie algebra g is solvable if its derived series goes down to zero that is $g^{(n)} = 0$ for some $n \in N$ **.** We remark that any a belian Lie algebra is solvable and any simple algebra is non solvable .The following proposition gives some facts about solvability .

10.2 Lemma[6]:

A Lie algebra g is solvable if and only if it satiates the chain condition .

10.3 Theorem[8]:

Let g be a solvable Lie algebra over k , let $v \neq \{0\}$ be a finite – dimensional vector space over k , the algebrai closure of k , let π be a homomorphism of g in to g I(v) .

Then there exists a vector $v \neq 0$ in v which is an eigenvector of all the members of $\pi(g)$.

10.4 Example:

Let g be the 2 - dimensional nonabelian Lie algebra . Then g is spanned by $e_1$and e_2 and D_g is spanned by e_1. This implies that $D_g^2 = \{0\}$, i,e, g is a solvable Lie algebra .

11. Definition (**Nilpotent Lie algebras**):

A lie algebra g over k is said to be nilpotent if for each $\mathrm{z} \in g$, $ad_g\mathrm{z}$ is a nilpotent endomorphism of g . A Lie group is called nilpotent if its Lie algebra is nilpotent .

11.1 Theorem[3]:

Let v be a nonzero finite - dimensional vector space over k and let g be a subalgebra of $g\mathrm{I}(v)$ consisting of nilpotent elements . Then

- **i.** g is nilpotent .
- **ii.** There exists a vector $v \neq 0$ in v such that $\mathrm{z}v = 0$ for all $\mathrm{z} \in g$.
- **iii.** There exists a basis $e_1,\ldots e_n$ of v in terms of which all the endomorphisms $x \in g$ are expressed by matrices with zeros on and below the diagonal .

11.2 Example:

Let $M_n(k)$ be the lie algebra of $n \times n$ matrices with entries in k . let $n(n,k)$ be the Lie sub algebra of all upper triangular marries in $M_n(k)$ with zeros on the diagonal . Then $n(n,k)$ is anipotent lie algebra.

11.3 Engel'sTheorem[3]

If all elements of g are **ad-nilpotent**, then g is nilpotent. This theorem is very important because it helps us to show that a Lie algebra is nilpotent without directly calculating its descending **central series.**

12. Root systems:

Before we introduce root systems of Lie algebras we give some preliminary notions which help in understanding the required ideas . Also it is worth mentioning that root systems are very effective tools which are used in classifying and studying the structure of Lie algebras .

12.1 Definition (Root systems):

Let V be a real finite-dimensional vectors space and $R \subset V$ a finite set of nonzero vectors , R is called a root systems in V (and its memberscalled roots) if

(i) R generates V.

(ii) For each $\propto \in R$ there exists a reflection $S_{\propto}$ along $\propto$ leaving R invariant .

(iv) For all $\propto, \beta \in R$ the number $a_{\mathcal{B},\propto}$ determined by $s_{\propto}\beta = \beta - a_{\mathcal{B},\propto} \propto$ is an integer that is $a_{\mathcal{B},\propto} \in$ z. (the set of integers)

12.2 Theorem[5]:

Every root system has a set of simple roots such that for each $\alpha \in \Phi$ may be written as

$$\alpha = \sum_{\delta \in \Delta} k_\delta \delta,$$

With $K_\partial \in$ z and each K_∂ has same sign .

12.3 Example:

The following example is of the root system B_2 Where α and β below form a base for B_2

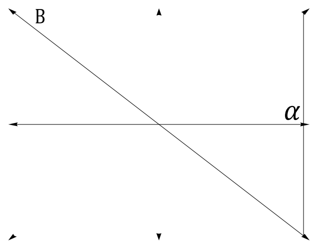

Figure (1.1)

The roots of a given basis are called simple.

13. The cartan Matrix:

For aroot system $\phi = \alpha_i, \alpha_j,\ldots$ one may define amatrix C by $C_{ij} = \langle \alpha_i, \alpha_j \rangle$ .This is the **Cartan matrix** of ϕ . Clearly the Cartan matrix is not symmetric however, Cartan matrices do possess several immediately observable and distinctive features. Forexample the main diagonal always consists of 2's and off-diagonal entries are restricted to integers of absolute value ≤ 3.

13.1 Definition(The generalizedCartan matrix):

A **generalized Cartan matrix** $\boldsymbol{A} = A_{ij}$ is asquare matrix with

integral entries such that.

(1) For non-diagonal entries, $A_{ij} \leq 0$.

(2) $A_{ij} = 0$ if and only if $A_{ji} = 0$

(3) A can be written as DS where D is adiagonal matrix and S is a symmetric matrix.

13.2 Example:

The Cartan matrix for the root system B_2 introduced previously, has the following form: $\begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}$.

14. Characterization of ϕ :

As a consequence of the transitive action of the Weylgroup on bases, it may be shown that the Cartan matrix of aroot system ϕ isindependent of the base chosen.

15 Dynkin diagram:

15.1 Definition (Dynkin diagram)

Let Δ be the set of simple roots of a root system Φ then we construct the Dynkin diagram of Δ in the following description:

- Each simple root α_i is represented by a circle or a dot as a vertex in Dynkin diagram.
- For each pair of simple roots $\alpha_i \neq \alpha_j$, we connect the corresponding vertices by n edges, where n depends on the angle φ between the two roots:
  - For $\varphi = \pi/2, n = 0$, the vertices are not connected ,and the case $A_1 \times A_1$ is diagram.
  - For $\varphi = 2\pi/3$, $n = 1$, the case is A_2 diagram with a single edge.
  - For $\varphi = 3\pi/4$, $n = 2$ the case is B_2 diagram with a double edge.
  - For $\varphi = 5\pi/6$, $n = 3$ the case is G_2 diagram with a triple edge.
- If, $\alpha_i \neq \alpha_j, |\alpha_i| \neq |\alpha_j|$ and $(\alpha_i , \alpha_j) \neq 0$ and, we orient the corresponding (multiple) edge by putting on it an arrow pointing towards the shorter root.

15.2 Example

For rank two root systems, see their Dynkin diagrams below:

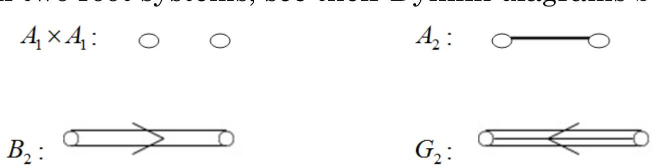

Figure (1.2)

All cases of Dynkin diagrams and the complete classification of Lie algebras by now we reach some general results which pave the way for researchers to carry their job . In addition, the problem of classification of symmetric spaces using the root systems of Lie algebras can be carried for more results and applications in other scientific fields.

16. The Weyl group

16.1 Definition (Weyl group)

The Weyl group is the group generated by the reflection $S_{\alpha,} \alpha \in \Delta$. Where $S_{\alpha,}$ is the reflection generated by the root α .

16.2 Theorem[10]

Given Δ and Δ^r bases of a root system $\Phi, \Delta^r = \sigma(\Delta)$ for some $\sigma \in W$.

17. (Homogeneous Space) :

17.1 Homogeneous Spaces of Lie Groups:

We consider the action of a Lie group on a manifold in special put important case , transitive action .

Let $\varphi: G \times M \to M$ denote such an action . There we recall that it is transitive if for every pair $p, q \in M$, there is a $g \in G$ such that

$$\varphi_g(p) = q.$$

This means that as far as properties preserved by G are concerned , any two points of the manifold are alike. A smooth manifold endowed with a transitive smooth action by a Lie group G is called a homogeneous G-space , or a homogeneous space or homogeneous manifold, if it is not important to specify the group. In most examples, the group action preserves some property of the manifold(such as distances in some metric, or class of curves such as straight lines in the plane); then the fact that the action is transitive means that the manifold looks likes the same everywhere from the point of view of this property . Often , homogeneous space are models for various kinds of geometric structures , and as such the play a central role in many areas of differential geometry.

17.2 Definition:

A manifold M is said to be a **homogeneous space** of the Lie group G if there is a transitive C^∞ action of G on M. Important example remains to be treated , since until this moment we have lacked an essential tool : Forbenius theorem. This example, viewed first from purely set theoretic stand point , is the following :

Let G be a group , H any subgroup, and G/H the set of left cossets. We define a left action $\lambda : G \times G/H \to G/H$ by $\lambda(g, xH) = gxH$; it is a left action since

(1) $\lambda(e, xH) = xH$ and

(2) $\lambda(g_1, \lambda(g_2\,, xH)) = \lambda(g_1\,, g_2xH) = (g_1g_2)xH = \ (g_1g_2, xH)$.

Moreover , if $\pi : G \to G/H$ is the natural mapping of each $g \in G$ to the coset which contains it , $\pi(g) = gh$ and if $L_g : G \to G$ denotes left translation , then we have the property:

(3) $\pi o L_g = \lambda_g o \pi$ (for all $g \in G$) .

The transitivity is apparent : $\lambda_{yx-1}(xH) = yH$ for all $x, y \in G$. Here are some important examples of homogeneous spaces .

17.3 Examples:

(1) The natural action of $O(n)$ on $S^{n-1}$is transitive , so is natural action of $SO(n)$ on $O(n)$ when $n \geq 2$. Thus for $n \geq 2$, S^{n-1} is a homogeneous space of either $O(n)$ or $SO(n)$.

17.4 Theorem[21](characterization of homogeneous spaces):

The mapping $\tilde{F} : G \to M$, defined by $\tilde{F} = \theta(g, a)$ is C^∞ and has rank equal to dim M everywhere on . The isotropy group H is a closed Lie subgroup, so that G/H is a C^∞manifold . The mapping $F :$ G/H → M defined by $F(gH) = \tilde{F}(g)$ is a diffeomorphism and $F o \lambda_g = \theta_g o F$ for every $H \in G$.

18. Symmetric Spaces :

18.1 Introduction:

Symmetric spaces are of great importance for several branches of mathematics. Any symmetric space has its own special geometry, such as Euclidean, elliptic and hyperbolic geometry. We can consider symmetric spaces from different points of view. We consider their algebraic features by considering Lie groups and their Lie algebras as algebraic approach to symmetric spaces. In fact a symmetric space can be considered as a Lie group G with a certain involution σ , or a homogeneous space $G/_H$ where G is a Lie group and H its isotropy subgroup.

18.2 Definition:

A Riemannian manifold (M, g) is said to be a symmetric space if for every point $p \in M$ their exists an isometry σ_p of (M, g) such that

(1) $\sigma_p(p) = p$, and

(2) $d\sigma_p = -id_{T_pM}$.

Such an isometry is called an involution at $p \in M$ .If M is any homogeneous space , i.e. its isometry group G acts transitively , then M is symmetric space if and only if there exists a symmetry σ_p .Namely , the symmetry at any other point $q = gp$ is just conjugate $\sigma_q = g\sigma_q g^{-1}$.

18.3 Symmetric Sub algebra :

If g is a compact simple Lie algebra ,σ is an involutive automorphism of g and $g = \mathfrak{h} \oplus p$ satisfying

i. $\sigma(X) = X , for\ X \in \mathfrak{h}, \sigma(X) = -X\ for\ X \in p$

$\mathfrak{h}$ is a subalgebra , but p is not ,and the following relations hold

ii. $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, [\mathfrak{h}, p] \subset \mathrm{p}, [p, p] \subset \mathfrak{h}$

A sub algebra $\mathfrak{h}$ satisfying (ii) is called symmetric sub algebra .

18.4 Theorem[4] :

Any symmetric space S determines a Cartan decomposition on the Lie algebra of Killing fields . Vise versa to any Lie algebra $g$with Cartan decomposition $g = \mathfrak{h} \oplus p$ there exists a unique simply connected symmetric space $S = G / H$ where G is the simply connected Lie group with Lie algebra g and H the connected subgroup with Lie algebra $\mathfrak{h}$.

19. Curvature:

19.1 Manifold of constant curvature

The manifold of constant curvature, it's a simplest Riemannian manifold.

19.2 Definition:

We recall that the Riemannian manifold M is said to have constant curvature if all sectional curvature at all points have the same constant value K. We suppose M to be a Riemannian manifold and let ω^i ,$1 \leq i \leq n$, denote the field of co-frames dual to an orthogonal frame field $E_1, \ldots, E_n$ on an open set $U \subset M$,with ω_i^j ,$1 \leq i, j \leq n$, denoting the corresponding connection forms . We then state the following lemma.

20 Basic properties:

The Cartan–Ambrose–Hicks theorem implies that $\boldsymbol{M}$ is locally Riemannian furthermore that any simply connected, complete locally Riemannian symmetric space is actually Riemannian symmetric. Any Riemannian symmetric space $\boldsymbol{M}$ is complete and Riemannian homogeneous (meaning that the isometry group of $\boldsymbol{M}$ acts transitively on $\boldsymbol{M}$). In fact, already the identity component of the isometry group acts transitively on $\boldsymbol{M}$ (because $\boldsymbol{M}$ is connected).

Locally Riemannian symmetric spaces that are not Riemannian symmetric may be constructed as quotients of Riemannian symmetric spaces by discrete groups of isometries with no fixed points, and as open subsets of (locally) Riemannian symmetric spaces.

21 Main Results:

- The elements of a Lie group can act as transformations on the elements of the symmetric spaces.
- Every Lie algebra corresponds to a given root system and each symmetric space corresponds to a restricted root system.
- The geometric and algebraic approaches to symmetric spaces can be modified to deduce each other.
- .Most of features of symmetric spaces can be extracted from Lie algebras.
- The study of symmetric spaces and continuous research in their properties and classification can lead to most surprising results that can help in their applications.
- We can have several different spaces derived from the same Lie algebra.
- The different approaches to symmetric spaces mentioned in this study lead to specifying two important features of symmetric spaces , that is Homogeneousness and symmetry .
- Homogeneousness can be considered as algebraic property through the transitive action of the isometry group while symmetry is a geometric property which can be seen through point reflection in the Riemannian manifold.
- Many properties of what is called symmetric spaces can be studied through their Lie algebras and root systems and this can help in extracting many of their properties.
- Root systems, Dynkin diagrams and Cartan matrix play an important role in classification of Lie algebras and they simplify this job.

- A finite root system can be encoded by Cartan matrix, which in turn can be encoded even more compactly by a Dynkin diagram.
- Dynkin diagrams correspond bijectively with finite-dimensional simple complex Lie algebras, and therefore the classification of Dynkin diagrams is actually a classification of all such Lie algebra.

References

[1] F. Dyson , Comm. Math.Phys.19(1970)235.
[2] M.A.Olshanetsky and A.M.Perelomov, Phys.Rep.94(1983)313.
[3] S . Helgason, Differential Geometry, Lie Groups & Symmetric Spaces (Academic Press , New York 1978) ISBN : 0-12-338460-5 .
[4] R .Gilmore, Lie Groups , Lie Algebras , and Some of Their Applications (John Wiley & Sons , New York 1974) ISBN : 0-471 – 30179 – 5 .
[5] W. M. Boothby, An Introduction to Differentiable Manifolds & Riemannian Geometry (Academic Press , New York 1975) .
[6] R . Hermann , Lie Groups for Physicists (W . A . Benjamin Inc. , NewYork 1966) .
[7] Ulrika Magnea, An Introduction to Symmetric Spaces , Dept. of Math. ,University of Torino 10 , 1 – 10125 , Italy .
[8] Stephanie Kernik , A very Brief Overview of Lie Algebras , University of Minnesota , Morris (2008) .
[9] Humphreys, James E. Introduction to Lie Algebras and Representation Theory, Springer, 1994.
[10] D. H. Sattinger, O. L. Weaver, Lie Groups & Lie Algebras