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Dynkin Diagrams, Root Systems and Cartan Matrix in Classification of Lie Algebras

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Abstract

In this paper we have introduced Lie algebras with some concentration on Lie algebras and root systems ,which help in classification and many applications of symmetric spaces . I address and explore the basic concept of a root system . First , its origins in the theory of Lie algebras are introduced and then an axiomatic definition is provided . Bases, Weyl groups, and the transitive action of the latter on the former are explained Finally , the Cartan matrix and Dynkin diagram are introduced to suggest the multiple applications of root systems to other fields of study and their classification.

Key words: Lie algebras, Root systems, Weyl group, Catan Matrix and Dynkin Diagrams

1. INTRODUCTION:

To anchor our discussion of root systems, let us begin with a general overview of their occurrence in the theory of Lie algebras. A Lie algebra may be understood as vector space with an additional bilinear operation known as the commutator [,] defined for all elements and satisfying certain properties. A lie algebra is called simple if its only ideals are itself and 0, and specifically the derived algebra $\{[x,y]\ /x\,,y \in L\} = [L,L] \neq 0$.(This is analogous to the commutator subgroup of a group being nontrivial). Let the Lie algebra L be semi simple ,i.e decomposable as the direct product of simple Lie algebras . Then we define a toral sub algebra as the span of some semi simple elements of L . It is natural to consider a maximal toral sub algebra H , which is not properly contained in any other. It turns out that L may then be written as the direct sum of H and the subspaces $L_\alpha = \{x \in L \setminus [h,x] = \alpha(h)x\ \forall\ h \in H\}$ where α ranges over all element of H^*. The nonzero α for which $L_\alpha \neq 0$ are called the roots of L relative to H. Root systems thus provide a relatively uncomplicated way of completely characterizing simple and semi simple Lie algebras. It is the goal of this paper to show that root systems may be themselves completely characterized by their Cartan matrices.

2. Lie algebra:

A Lie algebra is an algebraic structure whose main use is in studying geometric objects such as Lie groups and differentiable manifolds.

2.1 Definition:

A Lie algebra is a pair $(V,[\,,])$ where V is a vector space, and [,] is a Lie bracket, [,]: $V \times V \rightarrow V$ satisfying:

(1) $[v,w] = -[w,v]$ skew-symmetric.

(2) $[av + bu, w] = a[v,w] + b[u,w]$ a bilinear .

(3) $[v,[w,u]] + [w,[u,v]] + [u,[v,w]] = 0$For all $u\ and\ w \in V$ (Bianchi identity) . A Lie Bracket is a binary operation [,] on a vector space.

2.2 Example:

The Lie algebra $\boldsymbol{g}$ of R^n as a Lie group, is again $R^n$where $[X,Y\,] = 0 \forall\, X,Y\ \in G$

Thus, the Lie bracket for the Lie algebra of any abelian group is zero.

2.3 Example:

Let $V = R^3, [,]: R^3 \times R^3 \rightarrow R^3$as proved that it is a Lie algebra.

2.4 Example:

Let $\Omega(M)$ be the set of all vector fields on a manifold. **Define**

$$[v\,,w] = vw - wv\,,$$

Then $[v, w]$ is a Lie bracket. A homeomorphism of Lie algebra ℓ is a linear map, $\varphi : \ell \rightarrow \acute{\ell}$, preserving the bracket . This means that $\varphi[\ell_1, \ell_2] = [\varphi(\ell_1), \varphi(\ell_2)]$ for any $(\ell_1, \ell_2) \in \ell \times \ell$. A Lie subalgebra of Lie algebra ℓ is a sub-vector space η such that $[\eta, \eta] \subseteq \eta$. An ideal of ℓ is a Lie subalgebra η such that $[\eta, \ell] \subseteq \eta$

2.5 Definition:

A vector subspace η of a Lie algebra ℓ is called a Lie subalgebra if $[\eta, \ell] \subseteq \eta$.

2.6 Theorem[3] :

Let G be a Lie group and L its Lie algebra

(1) If H is a Lie subgroup of G , η is a Lie subalgebra of l .

(2) If η is a Lie sub algebra, there exists a unique Lie subgroup H of G such that the Lie algebra of H is isomorphic to η .

2.7Properties of a Lie algebra:

We now turn to the properties of a Lie algebra. These are derived from the properties of a Lie group. A Lie algebra has three properties:

(i) The operators in a Lie algebra form a linear vector space.

(ii) The operators closed under commutation: the commutator of two operators is in the Lie algebra;

(iii) The operators satisfy the Jacobi identity.

3. Root systems:

3.1Roots (Introduction):

The Root or Root vectors of Lie algebra are the weigh vectors of its ad joint representation. Roots are very important because they can be used both to define Lie algebra and to build their representations We will see that Dynkin Diagrams are in fact really only away to encode information about roots .The number of Roots is equal to the dimension of Lie algebra which is also equal to the dimension of the ad joint representation ,therefore we can associate a Root to every element of the algebra .The most important things about Roots is that they allow us to move from one weight to another . (Weights are vectors, which contain the eigenvalues of elements of Cartansubalgebra)

3.2 Definition (Root systems):

Let V be a real finite-dimensional vectors space and $R \subset V$ a finite set of nonzero vectors, R is called a root systems in V (and its memberscalled roots) if

(i) R Generates$V$.

(ii) For each $\propto \in R$ there exists a reflection $S_{\propto}$ along $\propto$ leaving R invariant.

(iv) For all $\propto, \beta \in R$ the number $a_{\mathcal{B},\propto}$ determined by ${}_{S_{\propto}}\beta = \beta - a_{\mathcal{B},\propto} \propto$ is

An integer that is$a_{\mathcal{B},\propto} \in$ z.

3.3 Theorem[4]:

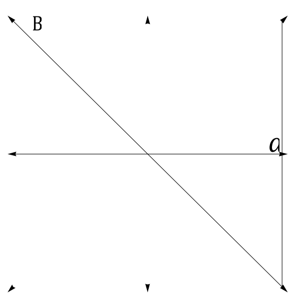
Every root system has a set of simple roots such that for each $\alpha \in \Phi$ may be written as

$$\alpha = \sum_{\delta \in \Delta} k_\delta \delta,$$

With $K_\partial \in$ z and each K_∂ has same sign .

3.4Example:

For convenience, we introduce the root system B_2 by way of providing an uncomplicated example for future reference. Note that α and ß as labeled form a base for B_2.



The roots which are part of a given basis are called **simple**. It follows from the simple roots' status as a basis that the **rank** of the base, i.e. the number of simple roots, is equal to the dimension of the Euclidean space E . The existence of such a base for any given root system may be proven in such a way that to determine an algorithm for finding a base given a root system. Let a root in ϕ be called indecomposable if it may not be written as a linear combination of any other roots. By selecting all the indecomposable roots whose inner product with a predetermined vector y in E is positive , one obtains a set of linearly independent roots a which lie entirely on the same side of the hyper plane normal to y. Then $-\alpha$ is not contained in the set for all α , and in fact these roots both span E and give rise to all other roots.

3.5 Definition (Reduced Root systems):

Let V be an Euclidean vector space (finite–dimensional real vector space with the canonical inner product$(.,.)$. Then $R \subseteq V \setminus \{0\}$ is reduced root systems if it has the following properties:

(1) The set R is finite and it contains a basis of the vector space V .

(2) For roots $\alpha , \beta \in R$ we demand $n_{\alpha,\beta}$ to be integer:

$$n_{\alpha,\beta} \equiv \frac{2(\alpha ,\beta)}{(\beta,\beta)} \in \mathrm{Z} .$$

(3) If $s_\alpha : V \longrightarrow V , S_\alpha(\lambda) = \lambda - \frac{2(\alpha,\lambda)}{(\alpha,\alpha)} \alpha \in \mathrm{Z}$.

(4) If $\alpha, c_\alpha \in R$ for some real c, then $c = 1 \quad or -1$

3.6 *Remark*[4]:

If α ,ß $\in R$ are porportional ,ß $= m\alpha$ ($m \in R$) then$m = \pm\frac{1}{2}$, $\pm 1, \pm 2$ in fact the numbers $a_{\alpha,m_\alpha} = 2 \backslash m$ and $a_{m_\alpha{}',\alpha} = 2\,m$ are both integers .A root system R is said to be reduced if α,ß $\in R$, ß $= m\,\alpha$ implies $m = \pm 1$. A root $\alpha \in R$ is called indivisible if $\frac{1}{2}\alpha \notin R$ and unmultipliable if $2\alpha \notin R$.

3.7 Example:

(i) The set $\Delta = \Delta(g , \mathfrak{h},)$of root asemisimple Lie algebra g over C with respect to
a cartan sub algebra $\mathfrak{h}$ is a reduced root system .

(ii) The set $\sum$ restricted roots is a root system which in general is not reduced.

3.8 Theorem[3]:

(i) Each root system has a basis.

(ii) Any tow bases are conjugate under a unique weyl group elements.

(iii) $a_{B,\alpha} \leq 0$ For any two different element $\alpha, \mathcal{B}^{/}$, in the same basis.

4. Weyl group:

The Weylgroup W of a root system consists of all the reflections σ_α generated by elements α of the root system . For a given root α , the reflection σ_α fixes the hyper plane normal to α and maps $\alpha \rightarrow -\alpha$. We may write $\sigma_\alpha(\beta) = \beta - \langle\alpha, \beta\rangle\, \alpha$. The hyper planes fixed by the elements of W partition E into Weyl chambers. For a given base Δ of E , the unique Weyl chamber containing all vectors y such that $(y, \alpha) \geq 0 \,\forall\, \alpha \in \Delta$ is called the fundamental Weyl chamber. We first prove the statement for $W^{/}$, the subgroup of W generated by only those rotations arising from the simple roots of a given base .

4.1 Theorem[4]:

Given Δ and $\Delta^{/}$ bases of a root system Φ, $\Delta^{/}\sigma(\Delta)$ for some $\sigma \in W^{/}$.

4.2 Lemma[1] :

For all $\alpha \in \phi$, $\exists\, \sigma \in W$ such that $\sigma(\alpha) \in \Delta$.

4.3 Lemma[1] :

$W^{/}$= { σ_α arising from $\alpha \in \Delta$} generates W.

5. The cartan Matrix:

For a root system $\phi = \alpha_i, \alpha_j$,... one may define a matrix C by $C_{ij} = \langle \alpha_i, \alpha_j \rangle$ .This is the **Cartan matrix** of ϕ . Clearly, the Cartan matrix is not symmetric however; Cartan matrices do possess several immediately observable and distinctive features. For example the main diagonal always consists of 2's and off-diagonal entries are restricted to integers of absolute value ≤ 3.

5.1 Definition (The generalized Cartan matrix):

A **generalized Cartan matrix** $\boldsymbol{A} = A_{ij}$ is asquare matrix with

integral entries such that.

(1) For non-diagonal entries, $A_{ij} \leq 0$.

(2) $A_{ij} = 0$ if and only if $A_{ji} = 0$

(3) A Can be written as DS where D is a diagonal matrix and S is a symmetric matrix.

5.2 Example:

The Cartan matrix for the root system B_2 introduced previously, has the following form: $\begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}$.

5.3Characterization of $\boldsymbol{\phi}$:

As a consequence of the transitive action of the Weylgroup on bases, it may be shown that the Cartan matrix of a root system ϕ isindependent of the base chosen.

5. 4 Theorem[4]:

Given two root systems $\phi \subset E$ and $\phi^{/} \subset E^{/}$ with bases

$\Delta = \{\alpha_i, \alpha_j, \ldots, \alpha_l\}$and $\Delta^{/} = \left\{ \alpha_i{}^{/}, \alpha_j{}^{/}\, \alpha_j{}^{-}, \ldots, \alpha_l{}^{/} \right\}$with identical cartan matrices i.e $\langle \alpha_i, \alpha_j \rangle = \langle\ \alpha_i{}^{/}, \alpha_j{}^{/}\ \rangle$ for $1 \leq i, j \leq 1$. Then this bijection extends to an isomorphism $f: E \longrightarrow E$. which maps

$$\phi \longrightarrow \phi^{/} \ \ and\ satisfies\ \langle f(\alpha), f(\beta) \rangle = \langle \alpha, \beta \rangle\ \forall\ \alpha, \beta \in \phi.$$

6 Dynkin Diagrams:

As mentioned previously, irreducible root systems provide a simple means of classifying Lie algebras. However, the root systems may themselves be classified according to their **Dynkin diagrams**. Each such diagrams belongs to one of finitely many families of graphs with avarietyof connections to e.g. Quiver representations. This correspondence between Cartan matrices and Dynkin diagrams may be explicitly under stood as follows. Each vertex of the Dynkin diagram correspond ds to a root α_i. Clearly if C_{ij}=C_{ji}= 0 , no edge exists between the vertices for $\alpha_i and\ \alpha_j$. If theC_{ij}th and C_{ji}th entries in the Cartan matrix are both ±1, a single edge connects the vertices corresponding to $\alpha_i\ and\ \alpha_j$.If the C_{ij}th or C_{ji}th entry is±2 or±3, two or three edges ,respectively , connect the two vertices in question .In order to distinguish the relative lengths of the roots , an arrow pointing towards the shorter of the two is drawn over the vertex in question .The properties of the Cartan matrices place a number of restrictions on possible. Dynkin diagrams ,which we enumerate below. In fact, these properties, enumerated below, lead to a complete description of all possible Dynkin diagrams, which may be found in [4] .

(1) If some of the vertices of the Dynkin diagram are omitted along with all their attached edges, the remaining graph is also possible as a Dynkin diagram.

(2) The number of vertex pairs connected by at least one edge is strictly less than the order of the root system. It follows that no Dynkin diagram may contain a cycle.

(3) No more than three edges can connect to a single vertex. Thus, the only Dynkin diagrams containing a triple edge contain exactly those two vertices it connects.

(4) If a Dynkin diagram contains as a sub graph a simple chain, the graph obtained by reducing that chain to a point also forms a Dynkin diagram. This prohibits several possible arrangements of terminal vertices from co-occurring with in a diagram , lest the preceding restriction be violated.

6.1 Definition (Dynkin diagram):

Suppose $s \subseteq R$ is a simple root system .The Dynkin diagram of s is a graph constructed by the following prescription.

(1) For each $\alpha_{i,} \in S$ we construct a vertex (visually, we draw a circle).
(2) For each pair of roots$\alpha_{i,,}\alpha_{j}$,, we draw a connection depending on the angle φ between them.

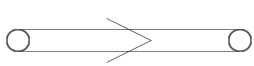
- If $\varphi =$ 90°the vertices are not connected, (we draw no line).
- If $\varphi =$ 120° the vertices have a single edge (we draw a single line) .
- If $\varphi = 135°$ the vertices have a double edge(we draw two connecting lines)
- If $\varphi = 150$ the vertices have a triple edge(we draw a three single connecting lines)

For double and triple edge connecting two roots, we direct them towards the shorter root (we draw an arrow pointing to the shorter root.

6.2 Example:

The Dynkin diagram for our familiar root system, B_2, is asfollows. Recall that e.g. $\beta$is longer than α.

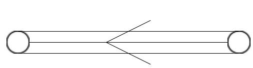
B_2



6.3Example:

The only Dynkin diagram with a triple edge, C_2, has the following form:

$\boldsymbol{C_2}$



7. MAIN RESULTS

- We can have several different spaces derived from the same Lie algebra
- Every Lie algebra corresponds to a given root system and each symmetric space corresponds to a restricted root system.
- Most of features of symmetric spaces can be extracted from Lie algebras.
- Many properties of symmetric spaces can be studied through their Lie algebras and root systems.

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