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Techniques for Solving a Certain Class of Partial Differential Equation by Fractional Fourier Transform

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Abstract

We have used the approach of Fourier transforms of fractional order, along the ordinary Fourier transform of first order. The integral form of the transform can be applied to build a table of fractional order Fourier transforms. A generalized operational calculus is analysed with the ordinary transform. The corresponding Green's functions are achieved in closed form. The given method is then applied to solve the problems using Green's, function. Its application gives a useful method for the solution of certain class of partial differential equations. The method of solution is discussed by its application to the second order partial differential equation, i.e, Heat Equation and Wave Equation.

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1. INTRODUCTION

The fractional Fourier transforms (FrFT) is a generalization of Fourier transforms which was introduced in the field of mathematics many years ago. (Namias, 1980) The idea of FrFT is to provide analytical solution to the Differential Equations of different classes, i.e., Partial Differential Equations. In recent times, the Fractional Fourier Transforms are being used in, communications and optics signal processing, due to its simple and effective properties.

1.1 Background Study

Due to the publication of (Condon, 1937) on the topic "Immersion of the Fourier Transform in continues group of functional transformation". Condon observes that "there exists a continuous group of functional transformations containing the Fourier Transforms as a subgroup". After 36 years, another publication (De-Bruijn, 1973) presented a 75 page research paper contains in its first few pages as FrFT. (Namias, 1980) used the term "fractional Fourier transform" in a more precise manner & found its application in quantum mechanics. (McBride & Kerr, 1987) modified and formulated aspects of the FrFT and used references of Condon, Bargman and Namias. Electrical engineers (Bradley Dickinson & Kenneth Stieglitz, 1982) work on eigenvectors of the discrete Fourier Transforms and apply it on signal processing.

1.2 Literature Review

In the span of time (1991 – 1993) two NASA mathematicians David Bailey and Paul Swarztrauber proposed fractional power of discrete Fourier Transforms. (Luis Almeida, 1993) used FrFT in time frequency to represent in signal processing. (David Mendlovic & Haldum Ozaktas, 1993) describe use of FrFT in an Optical system known as gradient index rod lens and till today FrFT is playing a key role in the area of optics, signal communication and the field of Physics and applied Mathematics.

The term fractional Fourier transforms is crucial in the study of Mathematics and physics. In the engineering systems, fractional Fourier transforms (generalized form of Fourier transforms). (De-Bruijn, 1973) introduced the term fractional Fourier transform for the first time and applied it in the field of generalized functions such as Wigner Distribution and Weyl Correspondence. (Namias, 1980) defined the concept of fractional Fourier transforms (FrFT) and used it as an application in the field of Quantum Mechanics. (McBride and Kerr, 1987) studied the Paper on (Namias, 1980) and modify the Namias's, Fractional operators. He also proved some theorems using modified Namias's, Fractional operators and applied to some Differential Equations.

(Kerr, 1988) developed some basic Solutions on Differential Equations using fractional powers of the Fourier Transform on the Schwartz space (Bailey and Swartzrauber, 1991) compute the Fractional Fourier transforms using Fast Fourier Transforms algorithm and discuss some properties of Discrete FrFT. (Almeida, 1994) suggested use of FrFT in the area of Communication in the time frequency plane. (Lin, 1999) investigated the use of FrFT in optics. (Ozaktas, kittay & Mendlovic, 1999) discussed the order fractional Fourier Transform which is a generalized from Ordinary Fourier transforms. (Bracewell, 2000) studied the Fourier analysis and the concerned are discussed. (Ozaktas, Zalevsky & kuttay, 2001) provide the concept of Fractional Fourier domains its properties and its relation to space-frequency representations. (Narayanan, & Prabhu, 2003) proposed some techniques in the field of signal restoration and noise removal by using FrFT. (Luchko, Martinez and Trujilo, 2008) introduced the new definition of Fractional Fourier Transforms using parameter ψ. (Roopkumar, 2016) used quaternion Fractional Fourier Transform techniques to prove Convolution and Product Theorem.

1.3 Definitions and Preliminaries

The Fourier transform is defined as:

$$u(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} v(x) e^{ikx} dx \tag{1.1}$$

and

$$v(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u(k) e^{-ikx} dk \tag{1.2}$$

The corresponding operator forms of above equations are:

$$F_{\frac{\pi}{2}}u(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} u(x')e^{ix'x}dx' \quad (1.3)$$

$$F_{-\frac{\pi}{2}}u(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} u(x')e^{-ix'x}dx' \quad (1.4)$$

The operators $F_{\frac{\pi}{2}}$ and $F_{-\frac{\pi}{2}}$ are complex conjugates. Then,

$$F_{\frac{\pi}{2}} . F_{-\frac{\pi}{2}} = F_{-\frac{\pi}{2}} . F_{\frac{\pi}{2}} = 1$$

Here we have noted that,

$$F_{\frac{\pi}{2}}u(x) = v(x), \qquad F_{\frac{\pi}{2}}v(x) = u(-x),$$
$$F_{\frac{\pi}{2}}u(-x) = v(-x), \qquad F_{\frac{\pi}{2}}v(-x) = u(x),$$

The corresponding Eigen-functions of the FT operator $F_{\frac{\pi}{2}}$ are $\exp\left(-\frac{x^2}{2}\right)H_n(x)$. We get $H_n(x)$ as "Hermite polynomials of order n". Then,

$$F_{\frac{\pi}{2}} \exp\left(-\frac{x^2}{2}\right)H_n(x) = e^{in\frac{\pi}{2}} \exp(-\frac{x^2}{2})H_n(x) \quad (1.5)$$

Consider $F_{\frac{\pi}{2}}$ which satisfy the Eigen values of equation,

$$F_{\psi} \exp\left(-\frac{x^2}{2}\right)H_n(x) = e^{in\psi} \exp(-\frac{x^2}{2})H_n(x) \quad (1.6)$$

It can be written as $F_{\psi} = e^{iA\psi}$. Then we have,

$$e^{iA\psi} \exp\left(-\frac{x^2}{2}\right)H_n(x) = e^{in\psi} \exp(-\frac{x^2}{2})H_n(x) \quad (1.7)$$

Taking derivative of (1.7), $\quad A\exp\left(-\frac{x^2}{2}\right)H_n(x) = n\exp(-\frac{x^2}{2})H_n(x)$

Since we know that,

$$H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0$$

From this relation we have following operator,

$$A = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 - \frac{1}{2} \quad (1.8)$$

The inverse FrFT is found by replacing ψ to $-\psi$, i.e., $F_{-\psi} = e^{-iA\psi}$. The Fourier transforms relates to $\psi = \frac{\pi}{2}$ and it's inverse to $\psi = -\frac{\pi}{2}$. The $\psi = 0$ takes the form of identity operator and $\psi = \pi$ shows the parity operator. If we explain the n^{th} order FrFT using $n = \frac{\psi}{(\pi/2)}$, we have the ordinary Fourier transform. The 0 order implies to Identity operator while order 2 implies to the parity transforms. Inverse transforms have negative orders. The range of transforms have ranged to $n = [-2,2]$. Transforms having non-integrals values of n termed as Fourier transform of fractional order. For FrFT of order $\frac{1}{2}$, it applied two times to obtain the ordinary Fourier transform. The transform of the operator $F_{\frac{\pi}{4}}$ may be termed as "square root of the ordinary Fourier transform".

Although the operator forms $\mathcal{F}_\psi = e^{iA\psi}$ helpful in theoretical consideration doesn't let itself to the straightforward calculation of the FrFT. In the case of ordinary Fourier transform, use of the operator $e^{iA\frac{\pi}{2}}$ is impracticable and one is place to integral form (1.1). Thus calculation of fractional transform can be obtained using corresponding integral form.

1.4 The Integral Form of Fractional Fourier Transform

We define it as,

$$\mathcal{F}_\psi e^{\left(-\frac{x^2}{2}\right)} H_n(x) = e^{in\psi} \exp(-\frac{x^2}{2})H_n(x)$$

elaborates the Hermite polynomials as the Eigen functions of the operator $\mathcal{F}_\psi$ with Eigen-values $e^{in\psi}$. We know that, any square integrable function u(x) can be written in Eigen functions, $u(x) = \sum_0^\infty \alpha_n e^{\left(-\frac{x^2}{2}\right)} H_n(x)$

Whereas α_n is the coefficient which can be determined as using perpendicularity of the Hermite polynomials:

$$\alpha_n = \frac{1}{2^n n!\sqrt{\pi}} \int_{-\infty}^{+\infty} u(x) \exp(-\frac{x^2}{2})H_n(x)dx \quad (1.9)$$

By applying $\mathcal{F}_\psi$ on (1.6), we have,

$$\mathcal{F}_\psi u(x) = \sum_0^\infty \alpha_n e^{in\psi} \exp(-\frac{x^2}{2})H_n(x) \quad (1.10)$$

The evaluation of transforms using series (1.10) is not practice generally. To find the integral form of $\mathcal{F}_\psi$, we use a formula due to, integral representation of the Hermite polynomials can be easily derived from Mehler's, formula (in Morse et al., 1953).

$$\sum_{n=0}^\infty \frac{e^{in\psi}}{2^n n!\sqrt{\pi}} H_n(x)H_n(x') = \frac{1}{\sqrt{\pi}\sqrt{1-e^{2i\psi}}} \exp[\frac{2xx'e^{i\psi} - e^{2i\psi}(x^2+x'^2)}{1-e^{2i\psi}}] \quad (1.11)$$

Substituting α_n from (1.9) into (1.10), and by Mehler's, formula,

$$\mathcal{F}_\psi u(x) = \frac{1}{\sqrt{\pi}\sqrt{1-e^{2i\psi}}} \int_{-\infty}^{+\infty} \exp\left[\frac{2xx'e^{i\psi} - e^{2i\psi}\left(x^2+x'^2\right)}{1-e^{2i\psi}}\right] \exp\left(-\frac{x^2}{2} - \frac{x'^2}{2}\right) u(x')dx'$$

(1.12)

Solving algebraically (1.12) expanded as:

$$\mathcal{F}_\psi u(x) = \frac{\exp i\left(\frac{\pi}{4} - \frac{\psi}{2}\right)}{\sqrt{2\pi\sin\psi}} \exp\left(-\frac{i}{2}\cot\psi x^2\right) \int_{-\infty}^{+\infty} \exp\left(-\frac{i}{2}\cot\psi x'^2 - \frac{ixx'}{\sin\psi}\right) u(x')dx' \quad (1.13)$$

The inverse transform is complex conjugate of $\mathcal{F}_\psi$ if ψ is real:

$$\mathcal{F}_{-\psi}u(x) = \frac{\exp\left[-i\left(\frac{\pi}{4}-\frac{\psi}{2}\right)\right]}{\sqrt{2\pi\sin\psi}} \exp\left(+\frac{i}{2}\cot\psi x^2\right) \int_{-\infty}^{+\infty} \exp\left(+\frac{i}{2}\cot\psi x'^2 - \frac{ixx'}{\sin\psi}\right) u(x')dx' \qquad (1.14)$$

We notice that when $\psi = \frac{\pi}{2}$ and $\psi = -\frac{\pi}{2}$, we again get the Fourier transforms (1.1) and (1.2). The transform shows to the identity operator if $\psi = 0$. Obviously, we convert sin ψ into ψ and cot ψ into $\frac{1}{\psi}$, when $\psi \to 0$, and [From generalized functions $\lim_{\varepsilon\to 0}\frac{1}{\sqrt{\varepsilon i\pi}} \exp\left(-x^2/i\varepsilon\right) = \delta(x)$] we can find that: $\int_{-\infty}^{+\infty} \delta(x-x')u(x')dx' = u(x)$.

In the same manner we can applied procedure when $\psi = \pi$, and it turns out to be the parity transformation

$$\mathcal{F}_{\pi}u(x) = \int_{-\infty}^{+\infty} \delta(x+x')u(x')dx' = u(-x).$$

In operator form, $\mathcal{F}_{\psi} = e^{iA\psi}$, we get that the product of two transforms with angles ψ and u, completed in progression, is equal to a transform with angle $\psi +$ u, that is $\mathcal{F}_{\psi+u} = \mathcal{F}_{\psi}\,\mathcal{F}_{u}$. This can be satisfied by the integral form. Derivation of FrFT integrals is much difficult and it doesn't show any measurement for its valid range. We notice that if ψ is real, the integrals showing the FrFT are associated with usual Fourier transforms of $u(x)e^{\left(\pm i\cot\psi\frac{x^2}{2}\right)}$ and we deduce that $u(x) \in L^2$ (Lebesgue class) on the interval $(-\infty, +\infty)$, so would the transform. When ψ has complex value the transform integral (1.14) vanish due to negative real part in the coefficient of x'^2.

We apply the integral form for calculation of FrFT of basic functions. Its calculations are easy and simple, only final formulas will be established in Table 1. In the next section, some basic rules will be derived, which can be used evaluated many transforms of fundamental functions. The FrFT is used for some specific non-square integerable functions (i.e. 1, x, x^2, etc.). If $\psi = \pm\frac{\pi}{2}$, some difficulties occur for example the case of the ordinary Fourier transform, but this may be avoided using delta function.

TABLE

Some simple functions and their fractional Fourier Transforms

Function u(x)	Fractional Fourier transform
$\exp\left(-\frac{x^2}{2}\right)$	$\exp\left(-\frac{x^2}{2}\right)$
$H_n(x)\exp\left(-\frac{x^2}{2}\right)$	$e^{in\psi}H_n(x)\exp\left(-\frac{x^2}{2}\right)$
$\exp\left(-\frac{x^2}{2}+ax\right)$	$\exp\left(-\frac{x^2}{2}-\frac{ia}{2}e^{i\psi}\sin\psi+axe^{i\psi}\right)$
$\delta(x)$	$\frac{\exp\left[i\left(\frac{\pi}{4}-\frac{\psi}{2}\right)\right]}{\sqrt{2\pi\sin\psi}}\exp\left(-\frac{i}{2}x^2\cot\psi\right)$
$\delta(x-a)$	$\frac{\exp\left[i\left(\frac{\pi}{4}-\frac{\psi}{2}\right)\right]}{\sqrt{2\pi\sin\psi}}\exp\left(-\frac{i}{2}(x^2+\psi^2)\cot\psi+iax\mathrm{cosec}\psi\right)$
1	$\frac{\exp\left[i\left(-\frac{\psi}{2}\right)\right]}{\sqrt{\cos\psi}}\exp\left(\frac{ix^2}{2}\tan\psi\right)$
e^{ikx}	$\frac{\exp\left[i\left(-\frac{\psi}{2}\right)\right]}{\sqrt{\cos\psi}}\exp\left(\frac{i}{2}(x^2+k^2)\tan\psi+ikx\sec\psi\right)$

1.5 Operational Calculus of FrFT

An operational calculus of fractional transform is based on the same principles of ordinary Fourier and Laplace transforms.

1.5.1 The Product Principle

We want to find the FrFT of $x^m u(x)$, such that $u(x) \in L^2$, over the interval $(-\infty, +\infty)$. We have from the integral form (1.5), to find this result instantly from the Eigen-value equation (1.6). We know that,

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0,$$

We find,

$$\mathcal{F}_\psi x\exp\left(-\frac{x^2}{2}\right)H_n(x) = x\exp\left(-\frac{x^2}{2}\right)e^{i(n+1)\psi}H_n(x) + n\exp\left(-\frac{x^2}{2}\right)e^{i(n-1)\psi} - e^{i(n+1)\psi}H_{n-1}(x) \qquad (1.15)$$

On using the fact that $H'_n(x) = 2H_{n-1}(x)$, we find,

$$\frac{d}{dx}\mathcal{F}_\psi\, e^{\left(-\frac{x^2}{2}\right)}H_n(x) = -xe^{i\psi}\exp\left(-\frac{x^2}{2}\right)H_n(x) + 2n\, e^{i\psi}e^{\left(-\frac{x^2}{2}\right)}H_{n-1}(x) \qquad (1.16)$$

Eliminating $ne^{i\psi}\exp\left(-\frac{x^2}{2}\right)H_{n-1}(x)$ between equations (1.15) and (1.16), one obtains

$$Ƒ_{\psi} x\, e^{\left(-\frac{x^2}{2}\right)} H_n(x) = \left(x \cos\psi + \frac{1}{i} \mathrm{Sin}\psi \frac{d}{dx}\right) Ƒ_{\psi} e^{\left(-\frac{x^2}{2}\right)} H_n(x)$$

Due to the expansion of (1.6), we have,

$$Ƒ_{\psi}(xu) = \left(x \cos\psi + \frac{1}{i} \mathrm{Sin}\psi \frac{d}{dx}\right) Ƒ_{\psi} u \qquad (1.17)$$

In operator form, this can be shown as:

$$Ƒ_{\psi} x = \left(x \cos\psi + \frac{1}{i} \mathrm{Sin}\psi \frac{d}{dx}\right) Ƒ_{\psi} \qquad (1.18)$$

Repeated use of equation (1.18) yields

$$Ƒ_{\psi} x^m = \left(x \cos\psi + \frac{1}{i} \mathrm{Sin}\psi \frac{d}{dx}\right)^m Ƒ_{\psi} \qquad (1.19)$$

From equation (1.19),

$$Ƒ_{\psi}(x^2 u) = \frac{1}{2} \sin 2\psi(-i + x^2 \cot\psi)\, Ƒ_{\psi}(u) - i x \sin 2\psi \frac{d}{dx} Ƒ_{\psi}(u) - \sin^2\psi \frac{d^2}{dx^2} Ƒ_{\psi}(u) \qquad (1.20)$$

Assumed an expanded function v(x) in Taylor series $v(x) = \sum b_m x^m$.
From (1.19), we find the generalized operator form as,

$$Ƒ_{\psi} v(x) = v\left(x \cos\psi + \frac{1}{i} \mathrm{Sin}\,\psi \frac{d}{dx}\right) Ƒ_{\psi} \qquad (1.21)$$

Applying (1.21) to u(x), we find,

$$Ƒ_{\psi} uv = v\left(x \cos\psi + \frac{1}{i} \mathrm{Sin}\psi \frac{d}{dx}\right) Ƒ_{\psi} u \qquad (1.22)$$

By interchanging f and g, we have,

$$Ƒ_{\psi}(vu) = u\left(x \cos\psi + \frac{1}{i} \mathrm{Sin}\psi \frac{d}{dx}\right) Ƒ_{\psi} v \qquad (1.23)$$

1.5.2 The Differentiation Principle

We want to find the transform of the differentiation of a function. By changing u(x) by $\frac{du}{dx}$ in the integral form (1.5) and using integration by parts, taking $u(x) \to 0$, when $x \to \pm\infty$. Then,

$$Ƒ_{\psi}\left(\frac{du}{dx}\right) = i \cot\psi\, Ƒ_{\psi}(xu) - \left(\frac{ix}{\sin\psi}\right) Ƒ_{\psi}(u)$$

Using equation (1.17), we have,

$$Ƒ_{\psi}\left(\frac{du}{dx}\right) = \left(-ix \sin\psi + \cos\psi \frac{d}{dx}\right) Ƒ_{\psi} u \qquad (1.24)$$

In operator form,

$$Ƒ_{\psi}\left(\frac{d}{dx}\right) = \left(-ix \sin\psi + \cos\psi \frac{d}{dx}\right) Ƒ_{\psi} \qquad (1.25)$$

And we can find it for higher derivatives as:

$$Ƒ_{\psi}\left(\frac{d}{dx}\right)^m = \left(-ix \sin\psi + \cos\psi \frac{d}{dx}\right)^m Ƒ_{\psi} \qquad (1.26)$$

Applying (1.26), we find,

$$Ƒ_{\psi}\frac{d^2u}{dx^2} = -(x^2\sin\psi + i\cos\psi)\sin\psi\, Ƒ_{\psi}(u) - ix\sin2\psi\frac{d}{dx}Ƒ_{\psi}(f) + \cos^2\psi\frac{d^2}{dx^2}Ƒ_{\psi}(u) \quad (1.27)$$

Again, we have,

$$Ƒ_{\psi}\, v\left(\frac{du}{dx}\right) = v\left(-ix\sin\psi + \cos\psi\frac{d}{dx}\right) Ƒ_{\psi}u \quad (1.28)$$

1.5.3 The Mixed Product Principle

Using (1.17) and (1.24) we can easily find a product $x\left(\frac{du}{dx}\right)$. The result is

$$Ƒ_{\psi}\left(\frac{du}{dx}\right) = -(\sin\psi + ix^2\cos\psi)\sin\psi\, Ƒ_{\psi}(u) + x\cos2\psi\frac{d}{dx}Ƒ_{\psi}(u) - \frac{i}{2}\sin2\psi\frac{d^2}{dx^2}Ƒ_{\psi}(u) \quad (1.29)$$

Similarly in this way other mixed products can also be found.

1.5.4 The Quotient Principle

For $Ƒ_{\psi}\left(\frac{u}{x}\right)$, we have to start from (1.19), using $\frac{u}{x}$ in place of u.

$$Ƒ_{\psi}\left(\frac{u}{x}\right) = e^{\left(-\frac{ix^2}{2}\cot\psi\right)}\frac{\exp i\left(\frac{\pi}{4}-\frac{\psi}{2}\right)}{\sqrt{2\pi\sin\psi}}\int_{-\infty}^{+\infty}\exp\left(-\frac{ix'^2}{2}\cot\psi + ixx'\operatorname{cosec}\psi\right)\frac{u(x')}{x'}dx'$$

Multiplying above by $e^{\left(\frac{ix^2}{2}\cot\psi\right)}$, taking derivative with respect to x, and again multiplying the $e^{\left(-\frac{ix^2}{2}\cot\psi\right)}$. We have,

$$e^{\left(-\frac{ix^2}{2}\cot\psi\right)}\frac{d}{dx}\left[e^{\left(\frac{ix^2}{2}\cot\psi\right)}Ƒ_{\psi}\left(\frac{u}{x}\right)\right] = \left(\frac{i}{\sin\psi}\right)Ƒ_{\psi}u$$

From which,

$$Ƒ_{\psi}\left(\frac{u}{x}\right) = \left(\frac{i}{\sin\psi}\right)e^{\left(-\frac{ix^2}{2}\cot\psi\right)}\int_{-\infty}^{x}e^{\left(\frac{ix^2}{2}\cot\psi\right)}Ƒ_{\psi}u\,dx \quad (1.30)$$

This can also be find from (1.17).

1.5.5 The Integration Principle

From (1.24), we have,

$$Ƒ_{\psi}\left(\frac{dv}{dx}\right) = \left(-ix\sin\psi\, Ƒ_{\psi}v + \cos\psi\frac{d}{dx}Ƒ_{\psi}v\right)$$

Assuming $\left(\frac{dv}{dx}\right) = u(x)$, we have:

$$Ƒ_{\psi}\int_a^x u(x)\,dx = \left(-ix\sin\psi\, Ƒ_{\psi}\int_a^x u(x)\,dx + \cos\psi\frac{d}{dx}Ƒ_{\psi}\int_a^x u(x)\,dx\right)$$

To solve this O.D.E gives,

$$\mathcal{F}_{\psi} \int_a^x u(x)\, dx = \sec\psi \exp\left(-\frac{ix^2}{2}\tan\psi\right) \int_a^x \exp\left(\frac{ix^2}{2}\tan\psi\right) \mathcal{F}_{\psi} u\, dx \qquad (1.31)$$

1.5.6 The Shift Principle

We start again from (1.5), and by assuming $y = x' + m$, we obtain:

$$\mathcal{F}_{\psi} u(x+m) = \exp\left[-ik\sin\psi\left(x + \frac{m}{2}\cos\psi\right)\right] \mathcal{F}_{\psi} u_{[x+m\cos\psi]} \qquad (1.32)$$

Whereas bracketed index shows that the $\mathcal{F}_{\psi}(u)$ must be in $x + m\cos\psi$.

1.5.7 The Similarity Principle

A similarity principle having $\mathcal{F}_{\psi} u(ax)$ may found, but it doesn't have the simplicity of the corresponding result for the usual Fourier transform. Simply we have:

$\mathcal{F}_{\psi} u(-x) = \mathcal{F}_{\psi-\pi} u(x)$.

1.6 Green Function

A function $G(x,t)$ is called a Green's function of a linear differential operator $L = L(x)$ operating on distributions inside a subset of the Euclidean space R^n. On a point, t, is a solution of $LG(x,t) = \delta(x-t)$ whereas δ is the Dirac delta function. This characteristic of a Green's function can be applied to solve differential equations having the form of $Lu(x) = f(x)$.

If the L's, kernel is nontrivial, then $G(x,t)$ becomes an ordinary function. But, practically, using some combination of symmetrical conditions, boundary value conditions, and other external imposed conditions will give us a unique Green's function. Green's functions can be described as a Green's function number using the type of boundary conditions satisfied. Further, in general, Green's functions are distributions, not necessarily real-valued functions. Green's function is a helpful tool in the solution of wave and diffusion equations. The Green's function of the Hamiltonian is a vital concept that relates to the density of states in quantum mechanics.

1.7 Green's functions for PDEs and FrFT

Green's function is an operator used to solve difficult ODE and PDE which may be unsolvable by other methods. Green's functions are widely used in electrodynamics and quantum field theory, where the relevant differential operators are often difficult or impossible to solve exactly but can be solved perturbatively. In field theory, the Green's function is mainly described as the propagator or two-point correlated function since it is related to the probability of measuring the field at one point and given that it is sourced at another point.

In our construction of Green's functions for the heat equation, Fractional Fourier transforms plays a vital role via the 'differentiation becomes multiplication' rule. We use Green's identities that enable us to construct Green's functions for Heat's equation and other PDE's.

2. SOLUTION of PDE by FrFT

2.1 Solution of Heat Equation Using FrFT

Consider the heat equation of the form

$$u_t = ku_{xx} \tag{2.1}$$

Solution: The required solution $u(x,t)$ can be found using Green's, function $G(x,x',t)$ with initial condition $G(x,x',0) = \delta(x-x')$.

$$U(x,t) = \int_{-\infty}^{+\infty} G(x,x',t)u(x',0)dx' \tag{2.2}$$

By taking FrFT of (2.1),

$$Ƒ_\psi \frac{\partial u}{\partial t} = kƑ_\psi \frac{\partial^2 u}{\partial x^2} \tag{2.3}$$

From Eq (1.8), we can deduce following expression from Hermite polynomials (Namias. V, (1980)):

$$A + \frac{1}{2} - \frac{1}{2}x^2 = -\frac{1}{2}\frac{\partial^2}{\partial x^2}$$

Where A is a Hermite operator taken from (Namias. V, (1980)).

$$\text{Or} \quad -2A - 1 + x^2 = \frac{\partial^2}{\partial x^2} \tag{2.4}$$

From (2.3) and (2.4), we have,

$$Ƒ_\psi \frac{\partial u}{\partial t} = kƑ_\psi(-2A - 1 + x^2)u$$

$$Ƒ_\psi \frac{\partial u}{\partial t} = k(x^2 - 2A - 1)Ƒ_\psi u \tag{2.5}$$

And from (2.5) $\frac{\partial}{\partial t}Ƒ_\psi u = k(x^2 - 2A - 1)Ƒ_\psi u$ (2.6)

We know that,

$$\frac{\partial}{\partial t}Ƒ_\psi u = \frac{\partial Ƒ_\psi}{\partial t} u + \frac{\partial u}{\partial t}Ƒ_\psi \tag{2.7}$$

Using Eq (2.6), (2.7) becomes,

$$[\frac{\partial F_\psi}{\partial t}u + \frac{\partial u}{\partial t}F_\psi] = k(x^2 - 2A - 1)F_\psi u \qquad (2.8)$$

From Integral representation of (Namias. V, 1980). We have,

$$F_\psi = e^{i\psi A}$$

$$\frac{\partial F_\psi}{\partial t} = i\frac{\partial \psi}{\partial t}Ae^{i\psi A} = i\frac{\partial \psi}{\partial t}AF_\psi$$

(2.8) becomes,

$$[i\frac{\partial \psi}{\partial t}AF_\psi u + \frac{\partial u}{\partial t}F_\psi] = k(x^2 - 2A - 1)F_\psi u$$

$$i\frac{\partial \psi}{\partial t}AF_\psi u + \frac{\partial u}{\partial t}F_\psi - k(x^2 - 2A - 1)F_\psi u = 0$$

$$F_\psi[\{i\frac{\partial \psi}{\partial t}A - k(x^2 - 2A - 1)\}u + \frac{\partial u}{\partial t}] = 0 \qquad (2.9)$$

To eliminate A from (2.9). Here we choose,

$$\frac{\partial \psi}{\partial t} = 2ik$$

Integrating partially from 0 to t

$$\psi = 2ikt|_0^t \Rightarrow \psi = 2ikt$$

Whereas $\psi = 0$ when $t = 0$.

(2.9) reduces to

$$-k(x^2 - 1)u + \frac{\partial u}{\partial t} = 0 \Rightarrow k(x^2 - 1)u = \frac{\partial u}{\partial t}$$

By partially integrating the above equation, we have,

$$ln u = k(x^2 - 1)t + ln g(x)$$

Here $ln g(x)$ is a constant of partial integration.

$$ln u = ln \exp\{k(x^2 - 1)t\} + ln g(x)$$

$$u = \exp\{k(x^2 - 1)t\}g(x) \qquad (2.10)$$

$$u(x,t) = \exp\{k(x^2 - 1)t\}g(x) \qquad (2.11)$$

At $\psi = 0$, $F_\psi = F_0$ is an identity operator.

Thus $G(x, x', 0) = \delta(x - x')$.

Then $u(x, x', 0) = \exp\{k(x^2 - 1)\ t\}\ g(x)\ \delta(x - x')$ (2.12)

The Green‘s, Function is

$$G(x, x', 0) = F_\psi\ u(x, x', 0) = F_{2ikt}\{e^{k(x^2-1)t}g(x)\delta(x - x')\} \qquad (2.13)$$

$$= e^{k(x^2-1)t}g(x)\ F_{2ikt}\,\delta(x - x') \qquad (2.14)$$

Then the required solution becomes

$$u(x,t) = \int_{-\infty}^{+\infty} G(x, x', t)\ u(x', 0)dx' = e^{kt}\int_{-\infty}^{+\infty} e^{k(tx^2)}\ g(x)F_{2ikt}\ \delta(x - x')\ u(x', 0)\ dx'$$

2.2 Solution of Wave Equation Using FrFT

Consider a wave equation of the form

$$u_{tt} = c^2 u_{xx} \tag{2.15}$$

The proposed solution u(x, t) can be found using Green's, function G(x, x′, t) with initial condition G(x, x′, 0) = δ(x − x′).

$$u(x,t) = \int_{-\infty}^{+\infty} G(x,x',t)u(x',0)dx' \tag{2.16}$$

By taking FrFT of (2.15),

$$\mathcal{F}_\psi \frac{\partial^2 u}{\partial t^2} = c^2 \mathcal{F}_\psi \frac{\partial^2 u}{\partial x^2} \tag{2.17}$$

We can deduce following expression from Hermite polynomials (Namias. V, (1980)):

$$A + \frac{1}{2} - \frac{1}{2}x^2 = -\frac{1}{2}\frac{\partial^2}{\partial x^2}$$

$$\text{Or} \quad -2A - 1 + x^2 = \frac{\partial^2}{\partial x^2} \tag{2.18}$$

From (2.17) and (2.18), we have,

$$\mathcal{F}_\psi \frac{\partial^2 u}{\partial t^2} = c^2 \mathcal{F}_\psi(-2A - 1 + x^2)u$$

$$\mathcal{F}_\psi \frac{\partial^2 u}{\partial t^2} = c^2(x^2 - 2A - 1)\mathcal{F}_\psi u \tag{2.19}$$

And from (2.19) $\frac{\partial}{\partial t}\mathcal{F}_\psi u = c^2(x^2 - 2A - 1)\mathcal{F}_\psi u$ (2.20)

We know that,

$$\frac{\partial}{\partial t}\mathcal{F}_\psi u = \frac{\partial \mathcal{F}_\psi}{\partial t} u + \frac{\partial u}{\partial t}\mathcal{F}_\psi$$

Differentiate w.r.t "t" partially

$$\frac{\partial^2}{\partial t^2}\mathcal{F}_\psi u = \frac{\partial \mathcal{F}_\psi}{\partial t}\frac{\partial u}{\partial t} + u\frac{\partial^2 \mathcal{F}_\psi}{\partial t^2} + \frac{\partial u}{\partial t}\frac{\partial \mathcal{F}_\psi}{\partial t} + \frac{\partial^2 u}{\partial t^2}\mathcal{F}_\psi$$

$$\frac{\partial^2}{\partial t^2}\mathcal{F}_\psi u = u\frac{\partial^2 \mathcal{F}_\psi}{\partial t^2} + 2\frac{\partial u}{\partial t}\frac{\partial \mathcal{F}_\psi}{\partial t} + \frac{\partial^2 u}{\partial t^2}\mathcal{F}_\psi \tag{2.21}$$

Using (2.21), (2.19) becomes,

$$u\frac{\partial^2 \mathcal{F}_\psi}{\partial t^2} + 2\frac{\partial u}{\partial t}\frac{\partial \mathcal{F}_\psi}{\partial t} + \frac{\partial^2 u}{\partial t^2}\mathcal{F}_\psi = c^2(x^2 - 2A - 1)\mathcal{F}_\psi u \tag{2.22}$$

From (Namias. V, (1980)), we have,

$$\mathcal{F}_\psi = e^{i\psi A}$$

$$\frac{\partial \mathcal{F}_\psi}{\partial t} = i\frac{\partial \psi}{\partial t}Ae^{i\psi A} = i\frac{\partial \psi}{\partial t}A\mathcal{F}_\psi$$

$$\frac{\partial^2}{\partial t^2}\mathcal{F}_\psi = iA[\frac{\partial^2 \psi}{\partial t^2}\mathcal{F}_\psi + \frac{\partial \mathcal{F}_\psi}{\partial t}\frac{\partial \psi}{\partial t}$$

$$\frac{\partial^2}{\partial t^2}\mathcal{F}_\psi = iA[\frac{\partial^2 \psi}{\partial t^2}\mathcal{F}_\psi + i\frac{\partial \psi}{\partial t}A\mathcal{F}_\psi\frac{\partial \psi}{\partial t}$$

$$\frac{\partial^2}{\partial t^2}\mathcal{F}_\psi = iA\frac{\partial^2 \psi}{\partial t^2}\mathcal{F}_\psi - A^2(\frac{\partial \psi}{\partial t})^2\mathcal{F}_\psi \tag{2.23}$$

From Eq (2.23), (2.22) becomes,

$$u\left[iA\frac{\partial^2\psi}{\partial t^2}F_\psi - A^2\left(\frac{\partial\psi}{\partial t}\right)^2 F_\psi\right] + 2iA\frac{\partial u}{\partial t}\frac{\partial\psi}{\partial t}F_\psi + \frac{\partial^2 u}{\partial t^2}F_\psi = c^2(x^2-2A-1)F_\psi u$$

$$uiA\frac{\partial^2\psi}{\partial t^2}F_\psi - A^2u\left(\frac{\partial\psi}{\partial t}\right)^2 F_\psi + 2iA\frac{\partial u}{\partial t}\frac{\partial\psi}{\partial t}F_\psi + \frac{\partial^2 u}{\partial t^2}F_\psi - c^2(x^2-2A-1)F_\psi u = 0$$

$$F_\psi\left[uiA\frac{\partial^2\psi}{\partial t^2} - A^2\left(\frac{\partial\psi}{\partial t}\right)^2 u + 2iA\frac{\partial u}{\partial t}\frac{\partial\psi}{\partial t} + \frac{\partial^2 u}{\partial t^2} - c^2(x^2-2A-1)u\right] = 0$$

$$F_\psi\left[\left\{iA\frac{\partial^2\psi}{\partial t^2} - A^2\left(\frac{\partial\psi}{\partial t}\right)^2 - c^2(x^2-2A-1)\right\}u + 2iA\frac{\partial u}{\partial t}\frac{\partial\psi}{\partial t} + \frac{\partial^2 u}{\partial t^2}\right] = 0 \quad (2.24)$$

To eliminate A from (2.23). Here we choose,

$$\frac{\partial^2\psi}{\partial t^2} = 2ic^2$$

Integrating partially from 0 to t

$$\frac{\partial\psi}{\partial t} = 2ic^2t\Big|_0^t \Rightarrow \frac{\partial\psi}{\partial t} = 2ic^2t$$

Again integrating partially from 0 to t:

$$\psi = ic^2t^2\Big|_0^t \Rightarrow \psi = ic^2t^2$$

Whereas $\psi = 0$ when $t = 0$.

Eq (2.24) reduces to

$$-A^2\left(\frac{\partial\psi}{\partial t}\right)^2 u - c^2(x^2-1)u + 2iA\frac{\partial u}{\partial t}\frac{\partial\psi}{\partial t} + \frac{\partial^2 u}{\partial t^2} = 0$$

$$-A^2(2ic^2t)^2u - c^2(x^2-1)u + 2iA(2ic^2t)\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} = 0$$

$$\frac{\partial^2 u}{\partial t^2} - 4Ac^2t\frac{\partial u}{\partial t} - [4A^2c^4t^2 + c^2(x^2-1)]u = 0 \qquad (2.25)$$

This is a second-order linear homogenous PDE with variable coefficients.

For the linearly independent particular solutions $u_1(x,t)$ and $u_2(x,t)$ of the above homogenous equation has a proposed solution:

$$u(x,t) = c_1u_1(x,t) + c_2u_2(x,t) \qquad (2.26)$$

Where c_1 and c_2 are arbitrary constants.

At $\psi = 0$, $F_\psi = F_0$ is an identity operator.

Thus $G(x,x',0) = \delta(x-x')$.

Then $u(x,x',0) = [c_1u_1(x,t) + c_2u_2(x,t)]\delta(x-x')$ (2.27)

The Green's, Function is

$$G(x,x',0) = F_\psi u(x,x',0) = F_{(ic^2t^2)}\{[c_1u_1(x,t) + c_2u_2(x,t)]\delta(x-x')\} \quad (2.28)$$

$$= [c_1u_1(x,t) + c_2u_2(x,t)]\,F_{(ic^2t^2)}\,\delta(x-x') \qquad (2.29)$$

Then proposed solution becomes $u(x,t) = \int_{-\infty}^{+\infty} G(x,x',t)\,u(x',0)\,dx'$

$$\mathrm{u(x,t)} = \mathrm{e}^{ikt} \int_{-\infty}^{+\infty} [c_1 u_1(x,t) + c_2 u_2(x,t)]\, \mathrm{F}_{(ic^2t^2)}\; \delta(\mathrm{x}-\mathrm{x}')\, \mathrm{u}(\mathrm{x}',0)\, \mathrm{dx}'$$

3. RESULTS AND DISCUSSION

Here, we have discussed the two different types of solutions to the Schrodinger equation given in (Namias, 1980) and PDE in this study. In (Namias, 1980), Namias solved time dependent and time independent Schrödinger equations of the form: $\frac{-\hbar}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}kx^2\psi = E\psi$ and $\frac{-\hbar}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}kx^2\psi = i\hbar\frac{\partial\psi}{\partial t}$ respectively. For the solution of these equations, Namias used operator $\frac{\partial^2}{\partial x^2} = x^2 - 2A - 1$ derived from Hermite polynomials. The solution of these equations is solved using properties and formulas of fractional Fourier transforms along with Green functions. These solutions are given in the form of integral. We have applied the same idea of Namias to the proposed heat and wave equations, i.e, $u_t = ku_{xx}$ and $u_{tt} = c^2u_{xx}$ respectively. And found that these equations also show the same kind of solutions in the form of integral. In the proposed solution, the order ψ of fractional Fourier transforms remain real. The proposed method can be applied to any type of heat equation as well as a wave equation.

4. CONCLUSIONS

Here we have analyzed the different properties of fractional Fourier transforms given in (Namias, 1980). Some results used in this study are taken from (Namias, 1980). Our result shows that the order of fractional Fourier transform remains real because transform depends on the order of fractional power. Also, we have applied transform to some basic applications of partial differential equations. Mainly, Heat and Wave Equations are discussed and their results are analyzed with the help of Green function in integral form of fractional Fourier transform. In the future, we would like to analyze more applications using fractional Fourier transform lies in different fields like optics, signal processing, and telecom engineering.

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