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Conformal Mapping as a Tool in Solving Some Mathematical and Physical Problems

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Abstract

The aim of this modest study was to shed some light on one of the useful tools of complex analysis, which is the method of conformal mapping (Also called conformal transformation). Conformal transformations are optimal for solving various physical and engineering problems that are difficult to solve in their original form and in the given domain. This work starts by introducing the meaning of a ''Conformal Mapping'', then introducing its basic Properties. In the second part, it deals with a set of various examples that explain the behavior of these mappings and show how they map a given domain from its original form into a simpler one. Some of these examples mentioned in this study showed that conformal transformations could be used to determine harmonic functions, that is, to solve Laplace's equation in two dimensions, which is the equation that governs a variety of physical phenomena such as the steady-state temperature distribution in solids, electrostatics and inviscid and irrotational flow (potential flow). Other mathematical problems are treated. All problems that are dealt with in this work became easier to solve after using this technique. In addition, they showed that the harmonicity of a function is preserved under conformal maps and the forms of the boundary conditions change accordingly.

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I. INTRODUCTION

A conformal mapping, also called a conformal transformation, or biholomorphic map, is a transformation that preserves angles between curves. A mapping by an analytic function is conformal at every point of the domain of definition where the derivative does not vanish. Conformal mappings are extremely important in complex analysis, as well as in many areas of physics and engineering [10].

It can also be said that a conformal mapping simplifies some solving processes of problems, mapping complex polygonal geometries and transforming them into simpler geometries, easily studied. These transformations became possible, due to the conformal mapping property to modify only the polygon geometry, preserving the physical magnitudes in each point of it [1].

Two researchers have worked in the field of conformal mappings, and they elucidated the advantages of conformal mappings over other advanced engineering skills. The work was to determine how the critical stress is spread with respect to the rupture angle using a conformal transformation. In their words, a conformal transformation has proved to be a good engineering tool to solve footing on slope problems, and they concluded that the critical normal stress distribution of footing on a slope is spread evenly along the slip surface with the mapping technique [2].

One of the applications in which a conformal mapping was used is the complex velocity potential of the flow of an ideal fluid. It was found that the complex velocity potential can be determined by solving a problem in either a horizontal or vertical strip [3].

Another application using a conformal mapping is the one used to solve the so-called third kind boundary-value problem of Laplace's equation. Where the work performed provides a new method for solving the complex electrostatic field boundary-value problem and realizing the visualization of that. It is a new way of solving the complex electrostatic field boundary-value problem [4].

In this study, the main purpose was to focus on the use of analytic functions when certain conditions are imposed on them. In particular, it aimed at elucidating the topic of conformal mappings. Various examples are given to show how conformal maps change given domains and help to solve some boundary-value problems, which are difficult to solve in their original domains.

Definition. [Conformal Mapping] [10]. Let $w = f(z)$ be a complex mapping defined in a domain D and let $z_0 \in D$. We say that $w = f(z)$ is conformal at z_0 if for every pair of smooth curves C_1 and C_2 in D intersecting at z_0 the angle between C_1 and C_2 at z_0 is equal to the angle between the image curves C_1' and C_2' at $f(z_0)$ in both magnitude and sense.

Preservation of Angles

The two curves intersect at (u_0, v_0), and the angle at which they intersect there, is the angle α between the two tangents there.

$$\therefore \alpha = \alpha_1 - \alpha_2,$$

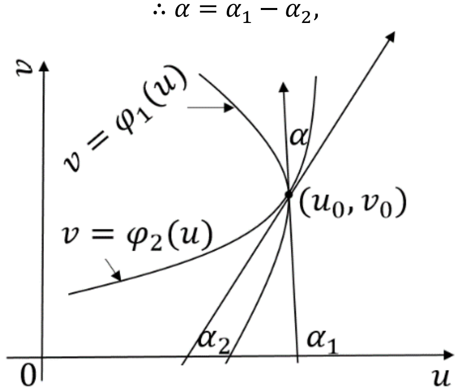

Figure 1: The w-plane.

where

$$\tan\alpha_1 = \left(\frac{dv}{du}\right)_1 = \varphi_1{}'(u),$$

$$\tan\alpha_2 = \left(\frac{dv}{du}\right)_2 = \varphi_2{}'(u).$$

Therefore $$\tan\alpha = \frac{\tan\alpha_1 - \tan\alpha_2}{1 + \tan\alpha_1 \tan\alpha_2} = \frac{\varphi_1'(u) - \varphi_2'(u)}{1 + \varphi_1'(u)\varphi_2'(u)},$$

evaluated when $u = u_0$.

If $w = u + iv = f(z)$ is analytic, then $v = \varphi_k(u), k = 1,2$ will have an image curve in the $z-plane$ which is given by $v(x,y) = \varphi_k\{u(x,y)\} \Rightarrow y = f_k(x), k = 1,2$.

If the two curves $v = \varphi_k(u), k = 1,2$, intersect at (u_0, v_0), then the curves $y = f_k(x), k = 1,2$, intersect at (x_0, y_0), where $u_0 = u(x_0, y_0), v_0 = v(x_0, y_0)$.

$$dv = d\varphi_k\{u(x,y)\},$$

$$\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = \varphi_k'(u)\left\{\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right\},$$

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}y'(x) = \varphi_k'(u)\left\{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}y'(x)\right\}.$$

Let $\frac{\partial u}{\partial x} = a$, $\frac{\partial u}{\partial y} = b$, $\frac{\partial v}{\partial x} = c$, $\frac{\partial v}{\partial y} = d$, therefore

$$\varphi_k'(u) = \frac{c + dy'(x)}{a + by'(x)} = \frac{c + df_k'(x)}{a + bf_k'(x)}.$$

But

$$a = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = d,$$

$$b = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -c,$$

Therefore

$$\varphi_k'(u) = \frac{af_k'(x) - b}{a + bf_k'(x)}.$$

Then $$\varphi_1'(u) - \varphi_2'(u) = \frac{(a^2+b^2)\left(f_1'(x) - f_2'(x)\right)}{\left(a + bf_1'(x)\right)\left(a + bf_2'(x)\right)}.$$

Also $$1 + \varphi_1'(u)\varphi_2'(u) = \frac{(a^2+b^2)\left(1 + f_1'(x)f_2'(x)\right)}{\left(a + bf_1'(x)\right)\left(a + bf_2'(x)\right)}.$$

$$\therefore \frac{\varphi_1'(u) - \varphi_2'(u)}{1 + \varphi_1'(u)\varphi_2'(u)} = \frac{(a^2+b^2)\left(f_1'(x) - f_2'(x)\right)}{(a^2+b^2)\left(1 + f_1'(x)f_2'(x)\right)} = \frac{f_1'(x) - f_2'(x)}{1 + f_1'(x)f_2'(x)} = \tan\alpha,$$

Provided that: $a^2 + b^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = {u_x}^2 + {v_x}^2 = |f'(z)|^2 \neq 0$.

$\therefore$ The curves $y = f_k(x), k = 1,2$ intersect at (x_0, y_0) at an angle α without any change provided that $f'(z_0) \neq 0$, $z_0 = (x_0, y_0)$.

Remark: Assume that $f(z)$ is analytic and non-constant in a domain D of the complex z-plane. If $f'(z_0) = 0$ for some $z_0 \in D$, then $f(z)$ is not conformal at this point. Such a point is called a Critical point of f [8].

Theorem. [Preservation of Harmonicity] If $w = u + iv = f(z) = f(x + iy)$ is a mapping, where $f(z)$ is analytic in a domain D of the z-plane and $\varphi(x, y)$ is harmonic in D ($\nabla^2\varphi = 0\ in\ D$), then its image $\varphi^*(x, y)$ is harmonic in D^* ($D \to D^*$).

Proof. To prove this, we see that

$$\varphi\{x(u,v), y(u,v)\} = \varphi^*(u,v) \Rightarrow d\varphi = d\varphi^*$$

$$\Rightarrow \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy = \frac{\partial\varphi^*}{\partial u}\left\{\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right\} + \frac{\partial\varphi^*}{\partial v}\left\{\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right\}.$$

Therefore,

$$\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi^*}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\varphi^*}{\partial v}\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial^2\varphi}{\partial x^2} = \frac{\partial\varphi^*}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial\varphi^*}{\partial v}\frac{\partial^2 v}{\partial x^2} + \frac{\partial u}{\partial x}\left\{\frac{\partial^2\varphi^*}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2\varphi^*}{\partial u\partial v}\frac{\partial v}{\partial x}\right\} + \frac{\partial v}{\partial x}\left\{\frac{\partial^2\varphi^*}{\partial u\partial v}\frac{\partial u}{\partial x} + \frac{\partial^2\varphi^*}{\partial v^2}\frac{\partial v}{\partial x}\right\}.$$

Therefore,

$$\frac{\partial^2\varphi}{\partial x^2} = \frac{\partial\varphi^*}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial\varphi^*}{\partial v}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2\varphi^*}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial^2\varphi^*}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + 2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\frac{\partial^2\varphi^*}{\partial u\partial v}.$$

$$\frac{\partial^2\varphi}{\partial y^2} = \frac{\partial\varphi^*}{\partial u}\frac{\partial^2 u}{\partial y^2} + \frac{\partial\varphi^*}{\partial v}\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2\varphi^*}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial^2\varphi^*}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2 + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\frac{\partial^2\varphi^*}{\partial u\partial v}$$

$$\therefore \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = \frac{\partial\varphi^*}{\partial u}(\nabla^2 u) + \frac{\partial\varphi^*}{\partial v}(\nabla^2 u) + \frac{\partial^2\varphi^*}{\partial u^2}\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right\} + \frac{\partial^2\varphi^*}{\partial v^2}\left\{\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right\}$$

$$+2\frac{\partial^2\varphi^*}{\partial u\partial v}\left\{\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right\}.$$

But if $f(z) = u(x,y) + iv(x,y)$ is analytic in D, then

$$\nabla^2 u = 0 = \nabla^2 v\ in\ D$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\ in\ D$$

$$\therefore \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = |f'(z)|^2\left(\frac{\partial^2\varphi^*}{\partial u^2} + \frac{\partial^2\varphi^*}{\partial v^2}\right).$$

Therefore, If $f'(z) \neq 0$ in D, then

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0\ in\ D \Rightarrow \frac{\partial^2\varphi^*}{\partial u^2} + \frac{\partial^2\varphi^*}{\partial v^2} = 0\ in\ D^*\ (i.e., \qquad \varphi^*(u,v)\ is\ harmonic\ in\ D^*).$$

Preservation of Boundary Conditions

Suppose that $w = u + iv = \zeta(z)$ is analytic in a domain D and $\varphi(x, y)$ is harmonic in D.

Suppose that ∂D has an equation $y = f(x)$. On ∂D the boundary condition is

$$\frac{\partial\varphi}{\partial n} = 0 = \underline{\nabla\varphi} \cdot n, \qquad where\, n = (\sin\alpha, -\cos\alpha).$$

$$\therefore \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}\right) \cdot (\sin\alpha, -\cos\alpha) = 0$$

$$\Rightarrow \frac{\partial\varphi}{\partial x}\tan x = \frac{\partial\varphi}{\partial y} \Rightarrow \frac{\partial\varphi}{\partial x} f'(x) = \frac{\partial\varphi}{\partial y} \ on\ D.$$

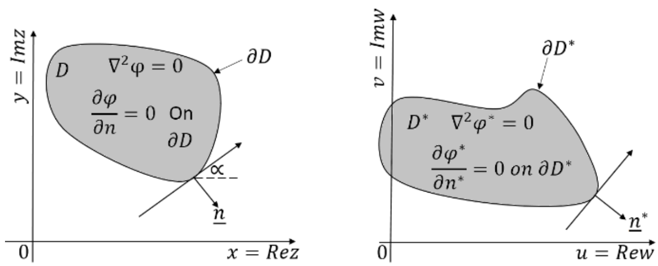

Figure 2: Transformation of Neumann condition.

In the w-plane we have the following configuration:
The equation of ∂D^* is given by $v = F(u)$, therefore $v(x,y) = F\{u(x,y)\}$ is the equation of ∂D.

$$\Rightarrow v(x,y) = F\{u(x,y)\} \Rightarrow y = f(x)$$

$$dv = dF$$

$$\therefore \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = F'(u)\left\{\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right\}$$

$$\Rightarrow \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx} = F'(u)\left\{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}\right\}$$

$$\therefore \frac{\frac{\partial v}{\partial x} - F'(u)\frac{\partial u}{\partial x}}{F'(u)\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}} = \frac{dy}{dx} = f'(x)$$

$$\therefore f'(x) = \frac{-v_x + u_x F'(u)}{F'(u)v_x + u_x}.$$

Now $\varphi(x,y) = \varphi\{x(u,v), y(u,v)\} = \varphi^*(u,v)$

$$\frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy = \frac{\partial\varphi^*}{\partial u}\{u_x dx + u_y dy\} + \frac{\partial\varphi^*}{\partial v}\{v_x dx + v_y dy\}$$

$$\therefore \frac{\partial\varphi}{\partial x} = \frac{\partial\varphi^*}{\partial u}u_x + \frac{\partial\varphi^*}{\partial v}v_x,$$

$$\frac{\partial\varphi}{\partial y} = -\frac{\partial\varphi^*}{\partial u}v_x + \frac{\partial\varphi^*}{\partial v}u_x.$$

$\therefore$ In the w-plane, the boundary condition on ∂D^* becomes

$$\frac{\left\{\frac{\partial\varphi^*}{\partial u}u_x+\frac{\partial\varphi^*}{\partial v}v_x\right\}\{u_xF'(u)-v_x\}}{u_x+v_xF'(u)}=u_x\frac{\partial\varphi^*}{\partial v}-v_x\frac{\partial\varphi^*}{\partial u}$$

$$\Rightarrow\left\{\frac{\partial\varphi^*}{\partial u}u_x+\frac{\partial\varphi^*}{\partial v}v_x\right\}\{u_xF'(u)-v_x\}=\left(u_x\frac{\partial\varphi^*}{\partial v}-v_x\frac{\partial\varphi^*}{\partial u}\right)(u_x+v_xF'(u)).$$

$$\therefore \frac{\partial\varphi^*}{\partial u}u_x^2F'(u)-\frac{\partial\varphi^*}{\partial u}u_xv_x+\frac{\partial\varphi^*}{\partial v}v_xu_xF'(u)-\frac{\partial\varphi^*}{\partial v}v_x^2=\frac{\partial\varphi^*}{\partial v}u_x^2+\frac{\partial\varphi^*}{\partial v}v_xu_xF'(u)-\frac{\partial\varphi^*}{\partial u}u_xv_x-\frac{\partial\varphi^*}{\partial u}v_x^2F'(u)$$

$$\Rightarrow\frac{\partial\varphi^*}{\partial u}u_x^2F'(u)-\frac{\partial\varphi^*}{\partial v}v_x^2=\frac{\partial\varphi^*}{\partial v}u_x^2-\frac{\partial\varphi^*}{\partial u}v_x^2F'(u)$$

$$\Rightarrow\frac{\partial\varphi^*}{\partial u}F'(u)(u_x^2+v_x^2)=\frac{\partial\varphi^*}{\partial v}(u_x^2+v_x^2)$$

$$\Rightarrow\left(\frac{\partial\varphi^*}{\partial u}F'(u)-\frac{\partial\varphi^*}{\partial v}\right)(|\zeta'(z)|^2)=0.$$

If $\zeta'(z)\neq 0$ on ∂D, then $\frac{\partial\varphi^*}{\partial u}F'(u)=\frac{\partial\varphi^*}{\partial v}$ on ∂D^*, which means that $\frac{\partial\varphi^*}{\partial n^*}=0$ on ∂D^*.

II. RELATED PROBLEMS

The aim of this part is to show how we use conformal transformations in solving mathematical and physical problems.

Example: How to map the domain in the w-plane, which is outside the triangle shown in Fig. (3) and $Imw\geq 0$, onto the upper half of the z-plane:

First, we regard $D=\lim_{p\to\infty}D_1$ as shown in Fig. (4), $\theta_k\to 0$ as $p\to\infty$, $k=1,2$.

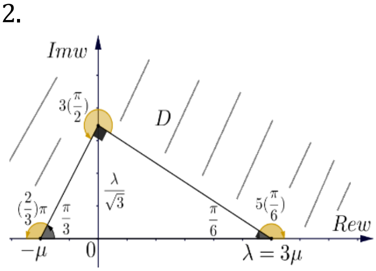

Figure 3: The w-plane.

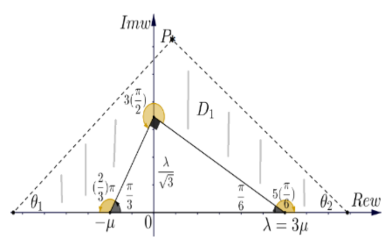

Figure 4: The w-plane.

$\frac{dw}{dz}=k(z+1)^{-\frac{1}{3}}z^{\frac{1}{2}}(z-a_3)^{-\frac{1}{6}},$

$$\therefore w = k \int_{z_0}^{z} \zeta^{\frac{1}{2}}(\zeta+1)^{-\frac{1}{3}}(\zeta-a_3)^{-\frac{1}{6}}d\zeta,$$

$$w = -\mu + k \int_{-1}^{z} \frac{\zeta^{\frac{1}{2}}d\zeta}{(\zeta+1)^{\frac{1}{3}}(\zeta-a_3)^{\frac{1}{6}}}.$$

When $w = i\mu\sqrt{3}$ we have $z = 0$

$$\therefore \mu\left(1+i\sqrt{3}\right) = k\int_{-1}^{0} \frac{\zeta^{\frac{1}{2}}d\zeta}{(\zeta+1)^{\frac{1}{3}}(\zeta-a_3)^{\frac{1}{6}}}. \quad \ldots\ldots. (1)$$

When $w = 3\mu$ we have $z = a_3 (a_3 > 0)$

$$\therefore 4\mu = k \int_{-1}^{a_3} \frac{\zeta^{\frac{1}{2}}d\zeta}{(\zeta+1)^{\frac{1}{3}}(\zeta-a_3)^{\frac{1}{6}}} = k \int_{-1}^{0} \frac{\zeta^{\frac{1}{2}}d\zeta}{(\zeta+1)^{\frac{1}{3}}(\zeta-a_3)^{\frac{1}{6}}} + k\int_{0}^{a_3} \frac{\zeta^{\frac{1}{2}}d\zeta}{(\zeta+1)^{\frac{1}{3}}(\zeta-a_3)^{\frac{1}{6}}}.$$

From (1) we have

$$4\mu = \mu\left(1+i\sqrt{3}\right) + k\int_{0}^{a_3} \frac{\zeta^{\frac{1}{2}}d\zeta}{(\zeta+1)^{\frac{1}{3}}(\zeta-a_3)^{\frac{1}{6}}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\therefore \frac{4\mu\sqrt{3}}{\sqrt{3}+i} = k \int_{0}^{a_3} \frac{\zeta^{\frac{1}{2}}d\zeta}{(\zeta+1)^{\frac{1}{3}}(\zeta-a_3)^{\frac{1}{6}}},$$

$$\Rightarrow \frac{4\mu\sqrt{3}}{2e^{i\frac{\pi}{6}}} = k \int_{0}^{a_3} \frac{\xi^{\frac{1}{2}}d\xi}{(\xi+1)^{\frac{1}{3}}\{e^{i\pi}(a_3-\xi)\}^{\frac{1}{6}}},$$

$$\Rightarrow 2\sqrt{3}\mu = k\int_{0}^{a_3} \frac{x^{\frac{1}{2}}dx}{(x+1)^{\frac{1}{3}}(a_3-x)^{\frac{1}{6}}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Also from (1) we have

$$\mu(1+i\sqrt{3}) = k \int_{-1}^{0} \frac{\zeta^{\frac{1}{2}}d\zeta}{(\zeta+1)^{\frac{1}{3}}(\zeta-a_3)^{\frac{1}{6}}},$$

On the path of integration we have $\zeta = te^{i\pi} \Rightarrow d\zeta = -dt, \zeta^{\frac{1}{2}} = it^{\frac{1}{2}}$

$$\therefore \mu(1+i\sqrt{3}) = k \int_{1}^{0} \frac{-it^{\frac{1}{2}}dt}{(1-t)^{\frac{1}{3}}\{e^{i\pi}(t+a_3)\}^{\frac{1}{6}}},$$

$$\therefore \frac{2e^{i\frac{\pi}{3}}e^{i\frac{\pi}{6}}\mu}{i} = k \int_{0}^{1} \frac{t^{\frac{1}{2}}dt}{(1-t)^{\frac{1}{3}}(t+a_3)^{\frac{1}{6}}},$$

$$\Rightarrow 2\mu = k\int_{0}^{1} \frac{\sqrt{t}\,dt}{(1-t)^{\frac{1}{3}}(t+a_3)^{\frac{1}{6}}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Equations (3) and (4) determine the constants k and a_3. They also show that k is real.

Note also that, the transformation can be written as an integral from $\zeta = 0$ to $\zeta = z$, as follows:

Since $$w = -\mu + k\int_{-1}^{0} \frac{\zeta^{\frac{1}{2}} d\zeta}{(\zeta+1)^{\frac{1}{3}}(\zeta-a_3)^{\frac{1}{6}}} + k\int_{0}^{z} \frac{\zeta^{\frac{1}{2}} d\zeta}{(\zeta+1)^{\frac{1}{3}}(\zeta-a_3)^{\frac{1}{6}}},$$

Using (1) we obtain

$$w = -\mu + \mu(1 + i\sqrt{3}) + k\int_{0}^{z} \frac{\zeta^{\frac{1}{2}} d\zeta}{(\zeta+1)^{\frac{1}{3}}(\zeta-a_3)^{\frac{1}{6}}},$$

Or $$w = i\mu\sqrt{3} + k\int_{0}^{z} \frac{\zeta^{\frac{1}{2}} d\zeta}{(\zeta+1)^{\frac{1}{3}}(\zeta-a_3)^{\frac{1}{6}}}.$$

Note that, in most of the cases, the Schwarz-Christoffel transformation is obtained in the form of an integral and cannot be obtained in closed form except in very few cases. However, the mapping can be used to find the behavior of w when z is very near to a certain it.

Put $z = a_3 + s, |z - a_3| \leq 1 \Rightarrow |s| \leq 1$ in $w - i\mu\sqrt{3} = k\int_{0}^{z} \frac{\zeta^{\frac{1}{2}} d\zeta}{(\zeta+1)^{\frac{1}{3}}(\zeta-a_3)^{\frac{1}{6}}}$,

$$\therefore w - i\mu\sqrt{3} = k\int_{0}^{a_3+s} \frac{\zeta^{\frac{1}{2}} d\zeta}{(\zeta+1)^{\frac{1}{3}}(\zeta-a_3)^{\frac{1}{6}}} = k\int_{0}^{a_3} \frac{\zeta^{\frac{1}{2}} d\zeta}{(\zeta+1)^{\frac{1}{3}}(\zeta-a_3)^{\frac{1}{6}}} + k\int_{a_3}^{a_3+s} \frac{\zeta^{\frac{1}{2}} d\zeta}{(\zeta+1)^{\frac{1}{3}}(\zeta-a_3)^{\frac{1}{6}}},$$

$$w - i\mu\sqrt{3} = \mu\left(3 - i\sqrt{3}\right) + k\int_{a_3}^{a_3+s} \frac{\zeta^{\frac{1}{2}} d\zeta}{(\zeta+1)^{\frac{1}{3}}(\zeta-a_3)^{\frac{1}{6}}}, \text{from Eqn. (2).}$$

Put $\zeta = a_3 + \tau$, hence

$$w - 3\mu = k\int_{0}^{s} \frac{(a_3+\tau)^{\frac{1}{2}} d\tau}{(a_3+1+\tau)^{\frac{1}{3}}\tau^{\frac{1}{6}}}.$$

$$\therefore w - 3\mu \sim k\frac{\sqrt{a_3}}{\sqrt[3]{1+a_3}}\frac{6}{5}(z-a_3)^{\frac{5}{6}}\ as\ z \to a_3.$$

Example: To map the domain shown in the diagram onto the upper half-plane:

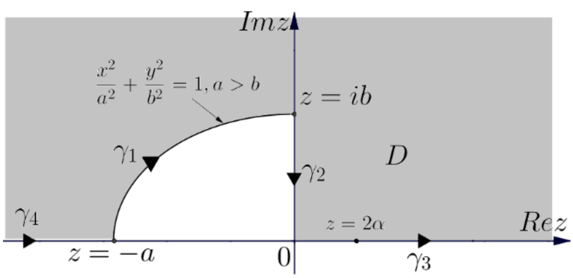

Figure 5: The z-plane.

1. First, we apply the conformal map

$$z = \alpha\left(\zeta + \frac{1}{\zeta}\right),$$

Where α is a real constant to be determined. And as $z \to \infty$ we have $\zeta \to \infty$.

To find γ_1^* (the image of γ_1 in the ζ −plane) we put

$$a\cos\theta + ib\sin\theta = \alpha\left\{Re^{i\varphi} + \frac{1}{R}e^{-i\varphi}\right\},$$

$$\Rightarrow \theta = \varphi, a = \alpha\left(R + \frac{1}{R}\right), b = \alpha\left(R - \frac{1}{R}\right), \left(\frac{\pi}{2} < \varphi < \pi\right)$$

$$\Rightarrow \frac{a+b}{2} = \alpha R, \frac{a-b}{2} = \frac{\alpha}{R} \Rightarrow \alpha^2 = \frac{a^2 - b^2}{4}$$

$$\Rightarrow \alpha = \frac{1}{2}\sqrt{a^2 - b^2}, \qquad R = \frac{a+b}{\sqrt{a^2 - b^2}} = \sqrt{\frac{a+b}{a-b}} = \rho > 1.$$

$\therefore \gamma_1^*$ Is the circular arc

$$\zeta = \rho e^{i\varphi}, \left(\frac{\pi}{2} < \varphi < \pi\right).$$

To find γ_2^* we put $z = iy$ $(y = b \to y = 0)$

$$\Rightarrow iy = \alpha\left(R + \frac{1}{R}\right)\cos\varphi + i\alpha\left(R - \frac{1}{R}\right)\sin\varphi$$

$$\Rightarrow \alpha\left(R + \frac{1}{R}\right)\cos\varphi = 0 \Rightarrow \varphi = \frac{\pi}{2},$$

and $$\frac{y}{\alpha}R = R^2 - 1 \Rightarrow \left(R - \frac{y}{2\alpha}\right)^2 = 1 + \frac{y^2}{4\alpha^2}.$$

$$\therefore R = \frac{y}{2\alpha} \pm \sqrt{1 + \frac{y^2}{4\alpha^2}}.$$

Because $R > 0$ we take $R = \frac{y}{2\alpha} + \sqrt{1 + \frac{y^2}{4\alpha^2}}$.

If $y = b$, then $R = \frac{b}{2\alpha} + \frac{a}{2\alpha} = \rho$. If $y = 0 \Rightarrow R = 1$.

$\therefore \gamma_2^*$ Is given by

$$\zeta = i\eta, (\eta = \rho \to \eta = 1).$$

To find γ_3^*: put $z = x > 0$

$$\Rightarrow \frac{x}{\alpha} = \zeta + \frac{1}{\zeta} \Rightarrow \left(\zeta - \frac{x}{2\alpha}\right)^2 = \frac{x^2}{4\alpha^2} - 1$$

$$\Rightarrow \zeta = \frac{x}{2\alpha} \pm \sqrt{\frac{x^2}{4\alpha^2} - 1}.$$

Because $\zeta \to \infty$ as $z \to \infty$ we must have $\zeta = \frac{x}{2\alpha} + \sqrt{\frac{x^2}{4\alpha^2} - 1}$.

If $0 < x < 2\alpha$, then

$$\zeta = \frac{x}{2\alpha} + i\sqrt{1 - \frac{x^2}{4\alpha^2}} \Rightarrow \xi^2 + \eta^2 = 1 = |\zeta|^2 \; and \; \zeta = e^{i\varphi}, \left(0 < \varphi < \frac{\pi}{2}\right).$$

If $x \geq 2\alpha$, then

$$\eta = 0, \xi = \frac{x}{2\alpha} + \sqrt{\frac{x^2}{4\alpha^2} - 1} \;\&\; \zeta = \xi, (1 \leq \xi < +\infty).$$

To find γ_4^* we put $z = x, x \leq -a$ or $z = -t$, $t \geq a$

$$-t = \alpha\left(\zeta + \frac{1}{\zeta}\right) \Rightarrow -\frac{t}{\alpha}\zeta = \zeta^2 + 1.$$

$$\therefore \left(\zeta + \frac{t}{2\alpha}\right)^2 = \frac{t^2}{4\alpha^2} - 1 \Rightarrow \zeta = \frac{-t}{2\alpha} \pm \sqrt{\frac{t^2}{4\alpha^2} - 1},$$

$\zeta \to \infty$ when $z \to \infty \Rightarrow \zeta = \frac{-t}{2\alpha} - \sqrt{\frac{t^2}{4\alpha^2} - 1}$.

Note that $t^2 - 4\alpha^2 > a^2 - (a^2 - b^2) = b^2$

$$\therefore 0 > \xi = \frac{-t}{2\alpha} - \sqrt{\frac{t^2}{4\alpha^2} - 1}, \eta = 0, -\rho \leq \xi < -\infty$$

$$\Rightarrow \zeta = Re^{i\pi}, (\rho \leq R < \infty).$$

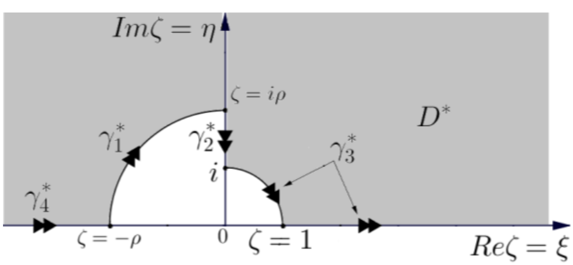

Figure 6: The ζ-plane.

2. Now, we apply the map $w = u + iv = \log\zeta$:

γ_1^{**} Is found by

$$w = u + iv = \log(\rho e^{i\varphi}) \Rightarrow u = \log\rho, v = \varphi\left(\frac{\pi}{2} < \varphi < \pi\right).$$

γ_2^{**} Is obtained by putting $w = \log\left(Re^{i\frac{\pi}{2}}\right) \Rightarrow u = \log R$ $(R = \rho \to R = 1)$, $v = \frac{\pi}{2}$ and for γ_3^{**} we put $w = \log(e^{i\varphi}) = i\varphi\left(0 < \varphi < \frac{\pi}{2}\right)$ and $w = \log R$ $(1 \leq R < \infty)$. For γ_4^{**} put $w = \log(Re^{i\pi})$ $(\rho \leq R < \infty)$.

Note that the map from D to D^{**} is given by

$$z = \alpha(e^w + e^{-w}) = 2\alpha\cosh w \Rightarrow z = \sqrt{a^2 - b^2}\cosh w.$$

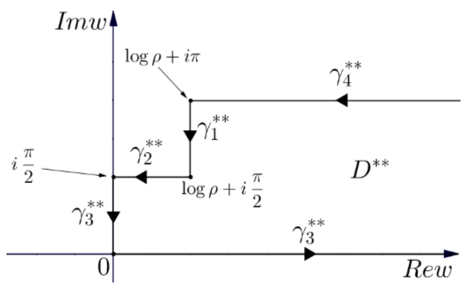

Figure 7: The w-plane.

To map D^{**} onto the upper half-plane we need the Schwarz-Christoffel transformation. This can be done by considering the limiting case as $P \to \infty$.

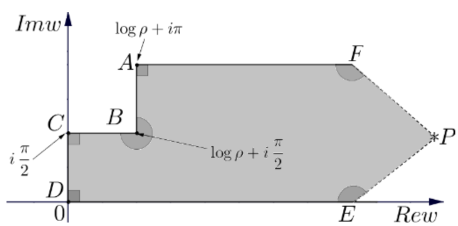

Figure 8: The w-plane.

Example: [The Electric Field Distribution in a Semi-Infinite Domain] [4,7]

$$\nabla^2\varphi = 0 \ in \ D = \{(x,y): y > 0, -\infty < x < \infty\}$$
$$\varphi = 0 \ on \ y = 0, -\infty < x < -1$$
$$\varphi = 1 \ on \ y = 0, 1 < x < \infty$$

The part $-1 < x < 1, y = 0$ is insulated ($i.e., \frac{\partial\varphi}{\partial y} = 0$).

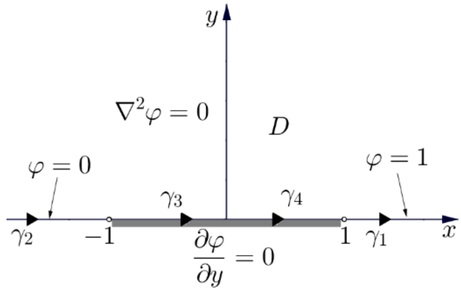

Figure 9: The z-plane.

This is a mixed boundary value problem, and there are two different boundary conditions on the same boundary line. It is difficult to find the electric potential distribution directly. In order to solve this boundary value problem easily, we consider the mapping

$$w = \sin^{-1}(z). \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The transformation function is

$$w = \sin^{-1}(z) = \frac{\pi}{2} + i\log\left\{z + \sqrt{z^2 - 1}\right\}.$$

On γ_1: let $z = 1 + t, (0 < t < \infty) \Rightarrow z - 1 = t \; and \; z + 1 = 2 + t.$

$\therefore \gamma_1^*$ Is given by

$$w = \frac{\pi}{2} + i\log\left\{1 + t + \sqrt{t(2+t)}\right\}.$$

$\therefore$ On γ_1^*:

$$u = \frac{\pi}{2} \; and \; v = \log\left\{1 + t + \sqrt{t(2+t)}\right\}, (0 < t < \infty)$$

$$\Rightarrow u = \frac{\pi}{2}, (0 < v < \infty).$$

On γ_2: let $z = (1+t)e^{i\pi}, (0 < t < \infty) \Rightarrow z + 1 = te^{i\pi} \; and \; z - 1 = (2 + t)e^{i\pi}$.

$\therefore \gamma_2^*$ is given by

$$w = \frac{\pi}{2} + i\log\left[-(1+t) - \sqrt{t(2+t)}\right]$$

$$= \frac{\pi}{2} + i\log\left[e^{i\pi}\{(1+t) + \sqrt{t(2+t)}\}\right]$$

$$= \frac{\pi}{2} + i\log\{e^{i\pi}\} + i\log\left\{1 + t + \sqrt{t(2+t)}\right\}$$

$$= \left(\frac{\pi}{2} - \pi\right) + i\log\left\{1 + t + \sqrt{t(2+t)}\right\}$$

$$= \left(-\frac{\pi}{2}\right) + i\log\left\{1 + t + \sqrt{t(2+t)}\right\}.$$

$\therefore$ On γ_2^*:

$$u=-\frac{\pi}{2}\ and\ v=\log\left\{1+t+\sqrt{t(2+t)}\right\},(0<t<\infty)$$
$$\Rightarrow u=-\frac{\pi}{2}\ and\ (0<v<\infty).$$

On γ_3: let $z=te^{i\pi},(0\le t<1)\Rightarrow z+1=1-t, z-1=(1+t)e^{i\pi}$.

$\therefore \gamma_3^*$ is given by

$$w=\tfrac{\pi}{2}+i\log\{-t+i\sqrt{(1-t^2)}\}=\tfrac{\pi}{2}+i\log\left\{\tfrac{-1}{t+i\sqrt{1-t^2}}\right\}$$
$$=\tfrac{\pi}{2}+i\log(e^{i\pi})-i\log\{t+i\sqrt{1-t^2}\}$$
$$=-\tfrac{\pi}{2}+\tan^{-1}\left\{\tfrac{\sqrt{1-t^2}}{t}\right\},(0\le t<1).$$

$\therefore$ On γ_3^*:

$$(-\frac{\pi}{2}<u\le 0)\ and\ v=0.$$

On γ_4: let $z=t,(0\le t<1)\Rightarrow z-1=(1-t)e^{i\pi}\ and\ z+1=1+t$.

$\therefore \gamma_4^*$ is given by

$$w=\frac{\pi}{2}+i\log\left(t+i\sqrt{1-t^2}\right)=\frac{\pi}{2}-\tan^{-1}\left\{\frac{\sqrt{1-t^2}}{t}\right\},(0\le t<1).$$

$\therefore$ On γ_4^*: $v=0\ and\ (0\le u<\frac{\pi}{2})$.

Thus, the upper half-plane of z-plane is mapped onto a semi-infinite strip of w-plan. The boundary condition at the bottom of the semi-infinite strip is the Neumann boundary condition ($i.e.$, $\frac{\partial\varphi^*}{\partial v}=0$), as shown

$$\begin{cases}\nabla^2\varphi^*=0\ in\ D^*=\left\{(u,v): v>0,-\frac{\pi}{2}<u<\frac{\pi}{2}\right\}\\ \varphi^*=0\ for\ 0<v<\infty, u=-\frac{\pi}{2}\\ \varphi^*=1\ for\ 0<v<\infty, u=\frac{\pi}{2}\end{cases}.$$

The part $-\frac{\pi}{2}<u<\frac{\pi}{2}, v=0$ is insulated ($i.e.,\frac{\partial\varphi^*}{\partial v}=0$).

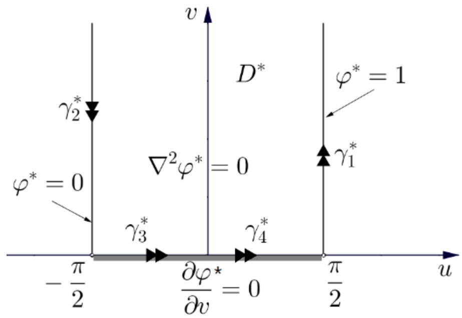

Figure10: The w-plane.

The electric field inside this domain is uniform, therefore, we should seek a solution that takes on constant values along the vertical lines $u = u_0$ and that $\varphi^*(u,v)$ should be a function of u alone. That is,

$$\varphi^*(u,v) = Au + B,$$

for some real constant A and B.
The boundary conditions above lead to

$$B = \frac{1}{2}, A = \frac{1}{\pi}.$$

$$\therefore \varphi^*(u,v) = \frac{1}{\pi}u + \frac{1}{2} = Re\left\{\frac{1}{2} + \frac{1}{\pi}w\right\}.$$

To find the solution of the original problem, we must substitute for u in terms of x and y as follows:

$$\frac{x^2}{\sin^2 u} - \frac{y^2}{\cos^2 u} = 1 \ \ldots\ldots\ldots \ (2)$$

Now, (2) is a hyperbola as shown

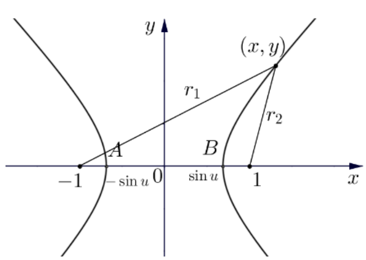

Figure11: The z-plane.

$AB = 2\sin u$. It is obvious that

$$r_1 + r_2 > 2\sin u. \ \ldots\ldots\ldots \ (3)$$

Put $\sin^2 u = t \Rightarrow \frac{x^2}{t} - \frac{y^2}{1-t} = 1 \Rightarrow x^2(1-t) - y^2 t = t - t^2.$

Therefore

$$\left\{t - \frac{(x^2+y^2+1)}{2}\right\}^2 = \frac{(x^2+y^2+1)^2 - 4x^2}{4}$$

$$\therefore 4t = 2x^2 + 2y^2 + 2 \pm \sqrt{\{x^2+y^2+1-2x\}\{x^2+y^2+1+2x\}}$$

$$= {r_1}^2 + {r_2}^2 \pm 2r_1r_2 = (r_1 \pm r_2)^2$$

$$\Rightarrow 4\sin^2 u = (r_1 \pm r_2)^2.$$

$$\therefore 2\sin u = r_1 \pm r_2.$$

In view of inequality (3), we reject the plus sign

$$\therefore \sin u = \tfrac{r_1 - r_2}{2}. \quad \ldots\ldots\ldots (4)$$

$$\therefore u = \sin^{-1}\left(\frac{r_2 - r_1}{2}\right),$$

or

$$u = \sin^{-1}\left\{\tfrac{\sqrt{(x+1)^2+y^2} - \sqrt{(x-1)^2+y^2}}{2}\right\}.$$

$\therefore$ The solution is

$$\varphi = \frac{1}{2} + \left(\frac{1}{\pi}\right)\sin^{-1}\left\{\frac{\sqrt{(x+1)^2+y^2} - \sqrt{(x-1)^2+y^2}}{2}\right\}.$$

Example: Use the appropriate transformation to solve the potential problem shown below:

$$\begin{cases} \nabla^2\varphi = 0 \ in \ D = \{z: |z| < 1\} \\ \varphi = 0 \ on \ \gamma_1 = \{z: |z| = 1, 0 < argz < \pi\} \\ \varphi = \varphi_0 \ on \ \gamma_2 = \{z: |z| = 1, \pi < argz < 2\pi\} \end{cases} .$$

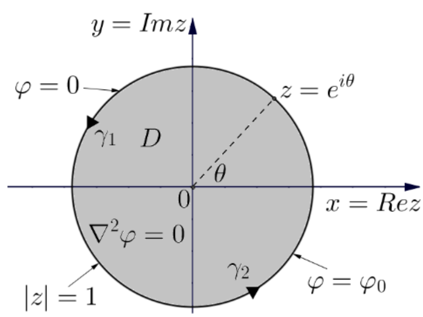

Figure12: The unit circle |z|=1 in the *z-plane*

Note that $\varphi(x,y)$ may be a temperature distribution in D or an electrostatic potential due to some distribution of electric charges or a velocity potential where D is a fluid region which is incompressible.

To solve the problem we employ the conformal transformation

$$z = \frac{w-i}{w+i}.$$

This mapping maps the upper half of the w-plane onto the unit disc (i.e., $D^* \to D$), on the other hand

$$zw + iz = w - i \Rightarrow w(1-z) = i(1+z),$$

$$\therefore w = u + iv = \frac{i(1+z)}{1-z}.$$

On γ_1: $z = e^{i\theta}, 0 < \theta < \pi$,

$$\Rightarrow w = u + iv = \frac{i\left(1+e^{i\theta}\right)}{1-e^{i\theta}} = \frac{i\left(e^{\frac{i\theta}{2}} + e^{\frac{-i\theta}{2}}\right)}{e^{\frac{-i\theta}{2}} - e^{\frac{i\theta}{2}}} = \frac{2i\cos\frac{1}{2}\theta}{-2i\sin\frac{1}{2}\theta} = \frac{-\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}.$$

$\therefore$ On γ_1^*: $v = 0, (-\infty < u < 0)$.

On γ_2: $z = e^{i\theta}, \pi < \theta < 2\pi$

$$\Rightarrow w = \frac{-\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}, \qquad \frac{\pi}{2} < \frac{\theta}{2} < \pi.$$

$\therefore$ on γ_2^*: $v = 0, 0 < u < \infty$.

By inspection, we see that

$$\varphi^* = \varphi_0\left\{1 - \frac{1}{\pi}\arg w\right\},$$

Or

$$\varphi^* = \varphi_0\left\{1 - \frac{1}{\pi}\tan^{-1}\frac{v}{u}\right\}.$$

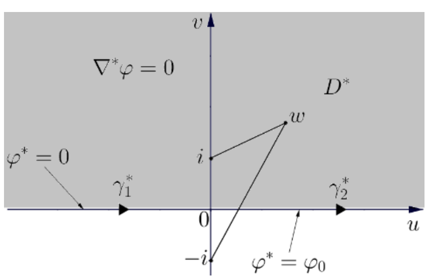

Figure13: The image of the circle in Fig. (11) in the $w - plane$.

$$\therefore \varphi = \varphi_0\left\{1 - \frac{1}{\pi}\tan^{-1}\frac{v(x,y)}{u(x,y)}\right\}.$$

Now

$$w = \frac{i(1+x+iy)}{1-x-iy} = \frac{i(1+x+iy)(1-x+iy)}{(1-x)^2+y^2} = \frac{-2y+i\left(1-x^2-y^2\right)}{(1-x)^2+y^2}$$

$$\therefore \varphi(x,y) = \varphi_0\left\{1 - \frac{1}{\pi}\tan^{-1}\frac{\left(1-x^2-y^2\right)}{-2y}\right\}.$$

III. CONCLUSIONS AND RECOMMENDATIONS

This part presents some conclusions derived from the conduct of the study of conformal mappings. It also provides some recommendations that can be followed when expanding the study.

CONCLUSION

From our modest study, we conclude the following:

- Conformal maps are transformations that preserve angles between curves (angles between tangents) including their orientation.
- Conformal maps are analytic functions whose derivatives do not vanish in the domains of definition (i.e., locally univalent).
- If f is a conformal map from D onto D^*, then f^{-1} (the inverse map) is also a conformal map from D^* onto D.
- The Schwarz-Christoffel transformation maps the interior of a polygon, say in the w-plane, onto the upper half of the z-plane.
- The harmonicity of a function is preserved under conformal maps.
- Conformal maps take complicated boundaries into simpler ones sometimes.
- We can determine a harmonic potential by using a conformal mapping that maps D onto D^* where the solution of the problem is easier to find.

RECOMMENDATIONS

This study dealt with the technique of conformal mapping without using computer programs and numerical techniques. Thus, the following recommendations are hereby presented:

- Since the study dealt with this technique without using computer programs, a study should be attempted using these programs.
- It is recommended that, numerical techniques should be used in the study of conformal maps.

REFERENCES

[1] Calixto, W. P., Alvarenga, B., da Mota, J. C., Brito, L. da C., Wu, M., Alves, A. J., … Antunes, C. F. R. L. (2010). Electromagnetic Problems Solving by Conformal Mapping: A Mathematical Operator for Optimization. Mathematical Problems in Engineering, 2010, 1–19. doi:10.1155/2010/742039

[2] Onyelowe, Ken C. and Agunwamba, J.C., Conformal mapping and Swartz-Christophel and transformation of the critical normal stress distribution of footing on slope, International journal of civil Engineering and Technology, Volume 3, Issue 1, January- June (2012), pp. 128-135.

[3] Mohammed Mukhtar Mohammed Zabih, R.M. Lahurikar, Applications of Conformal Mapping to Complex Velocity Potential of the Flow of an Ideal Fluid, International Journal of Mathematics Trends and Technology (IJMTT) – Volume 52 Number 3 December 2017.

[4] Fuqian Wang, Solution of the Third Kind Boundary Value Problem of Laplace's Equation Based on Conformal Mapping, Journal of Applied Mathematics and Physics, 2019, 7, 536-546.

[5] B. NYANDWI, "Applications of conformal mappings to the fluid flow", Ph.D. dissertation, University of Rwanda. 2018.

[6] D. Zill and P. Shanahan, A first course in complex analysis with applications, Jones & Bartlett Learning, 2003.

[7] J. H. Mathews, Basic complex variables for mathematics and engineering, Allyn & Bacon, 1982.

[8] S. Ganguli, "Conformal Mapping and its Applications," Department of Physics, University of Tennessee, Knoxville, TN37996, pp. 1-4, 2008.

[9] S. Madhwendra, "Application of Conformal Mapping: Electrostatic Potential", Galgotias University, 2020.

[10] https://mathworld.wolfram.com/ConformalMapping.html