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Insertion of Pythagorean fuzzification in PU-algebras

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Abstract

We infused the notion of Pythagorean fuzzy set in PU-algebra in this paper, and established several forms of Pythagorean fuzzy set in PU-algebra, explicitly Pythagorean fuzzy PU-sub algebra, Pythagorean fuzzy PU-ideals, and Pythagorean fuzzy new PU-ideals, and investigated certain of their useful properties. We have proved that a Pythagorean fuzzy set in Ψ *is a Pythagorean fuzzy ideal in PU – algebra, if* $\alpha_{\mathcal{P}}(ɒ)$ *is constant and* $\beta_{\mathcal{P}}(ɒ)$ *is identity for all* $ɒ \in \Psi$*. If* $\mathcal{P}$ $(\alpha_{\mathcal{P}}, \beta_{\mathcal{P}})$ *is supposed to be a Pythagorean fuzzy new-ideal of PU-algebra* Ψ*, and if the inequality* $ɒ * d \leq ɔ$ *holds in* Ψ*, then* $\alpha_{\mathcal{P}}(d) \geq \min\{\alpha_{\mathcal{P}}(ɒ), \alpha_{\mathcal{P}}(ɔ)\}$ *and* $\beta_{\mathcal{P}}(d) \leq \max\{\beta_{\mathcal{P}}(ɒ), \beta_{\mathcal{P}}(d)\}$*. Moreover we have proved that If every Pythagorean fuzzy sub-algebra A of PU – algebra* Ψ *satisfies the condition* $\alpha_{\mathcal{P}}(d * ɒ) \geq \alpha_{\mathcal{P}}(d)$ *and* $\beta_{\mathcal{P}}(d * ɒ) \leq \beta_{\mathcal{P}}(d)$ *then* $\alpha_{\mathcal{P}}$ *and* $\beta_{\mathcal{P}}$ *are constant functions.*

Keywords: PU-algebra; Pythagorean fuzzified sets; Pythagorean fuzzified PU-subalgebra; Pythagorean fuzzified PU- ideal; Pythagorean fuzzified PU- New ideal.

MSC 2010: 08A72, 03E72

1. INTRODUCTION

Algebras of logic are an important class of algebras for studying many algebraic structures. BCI - algebra [2], BCK - algebra [1], BE - algebra [3], PU - algebra [4], and topological structure of fuzzy PU - new ideal in PU - Algebra [5] are examples of these.

Pawlak [6] was the first to introduce the concept of rough sets in 1982. Ensuing the overview of the theory, several scholars worked on

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generalising the notion of rough set and applying it to numerous algebraic structures, for instance, Jun [7] and Dudek et al. [8] in 2002 smeared the rough set theory to BCI and BCK - algebras.

Zadeh [9] was the first to introduce the concept of fuzzy sets in the year 1965. The theories of fuzzification of a set formulated by Zadeh and others have established numerous applications in the field of mathematics and other disciplines. Pursuing Zadeh's [9] introduction of the concept of fuzzy sets, Atanas Sor [10] demarcated a new idea of intuitionistic fuzzified sets, which is a generalisation of fuzzy set. Satirad and Lampan [12] described a number of varieties of fuzzy sets and subsets of fully UP - semigroups, and scrutinized the algebraic properties of fuzzy sets with respect to nion and intersection. Yager [11] presented a different class of unusual fuzzy sub - sets caused by Pythagorean fuzzy subsets and the associated idea of Pythagorean membership grades.

The Pythagorean fuzzy set concept has been infused to semi-groups, ternary semi-groups and numerous logical algebraic structures for instance: Hussain et al. [13] introduce the idea of roughness of Pythagorean fuzzy ideals in a semigroup in 2019. This concept is then magnificated to upper and lower approximations of Pythagorean fuzzy right or left ideals, (1,2) – ideals, interior ideals, bi – ideals in a semigroup, and several significant properties associated with these theories are described. Jhansi and Mohana [14] presented and investigated the concept of bipolar Pythagorean fuzzy A-ideal of BCI - algebra. Also examined the correlation between bipolar Pythagorean fuzzy ideals, bipolar Pythagorean fuzzy ideals A - ideals and bipolar Pythagorean fuzzy subalgebras.

Chin Ram and Panityakul [15] investigated roughness of Pythagorean fuzzy ideals of ternary semigroups in 2020. This concept is protracted to Pythagorean fuzzy ideals' upper and lower approximations.

In the present manuscript, we inject the notion of Pythagorean fuzzification to PU – algebra and explore their properties. Moreover, we describe the association between the Pythagorean PU – ideals.

2- PRELIMINARIES

Before we go ahead, let's go over the definitions of basic terminologies of PU - algebra.

Definition 1.1: [4] A PU–algebra $(\Psi, \Theta, 0)$ is of a form of (2, 0) algebra fulfilling the **[Pu–1]** and **[Pu–2]** conditions for any ɒ, ɗ, ɔ $\in \Psi$. Where

[Pu–1] $0 \,\Theta$ ɒ = ɒ,

[Pu–2] (ɒ Θ ɔ) Θ (ɗ Θ ɔ) = ɗ Θ ɒ

On Ψ the relation " $\leq$ " is defined by ɒ $\leq$ ɗ $\Leftrightarrow$ ɗ Θ ɒ = 0

Example 1.2: Let we have a (R, Θ, 0) where R is a set of real number and Θ is defined by

ʊ Θ ʋ = ʋ - ʊ, ∀ ʊ, ʋ ∈ *R is a PU − algebra*.

Example 1.3: Let x = {0, ɒ, ʊ, ʋ, ɗ} in which Θ is defined by

Θ	0	ɒ	ʊ	ʋ	ɗ
0	0	ɒ	ʊ	ʋ	ɗ
ɒ	ɗ	0	ɒ	ʊ	ʋ
ʊ	ʋ	ɗ	0	ɒ	ʊ
ʋ	ʊ	ʋ	ɗ	0	ɒ
ɗ	ɒ	ʊ	ʋ	ɗ	0

Proposition 1.4: [4] The following conditions hold in any PU–algebra (Ψ, Θ, 0), ∀ ɒ, ɗ, ɔ ∈ Ψ.

(a) ɒ Θ ɒ = 0

(b) (ɒ Θ ɔ) Θ ɔ = ɒ

(c) ɒ Θ (ɗ Θ ɔ) = ɗ Θ (ɒ Θ ɔ)

(d) ɒ Θ (ɗ Θ ɒ) = ɗ Θ 0

(e) (ɒ Θ ɗ) Θ 0 = ɗ Θ ɒ

(f) If ɒ ≤ ɗ, then ɒ Θ 0 = ɗ Θ 0

(g) (ɒ Θ ɗ) Θ 0 = (ɒ Θ ɔ) Θ (ɗ Θ ɔ)

(h) ɒ Θ ɗ ≤ ɔ if and only if ɔ Θ ɗ ≤ ɒ

(i) ɒ ≤ ɗ if and only if ɗ Θ ɔ ≤ ɒ Θ ɔ

(j) In any PU–algebra (Ψ, Θ, 0), the following results are equivalent.

(1) ɒ = ɗ (2) ɒ Θ ɔ = ɗ Θ ɔ (3) ɔ Θ ɒ = ɔ Θ ɗ

(k) The left and right cancellation properties hold in Ψ.

Proposition 1.5 [4] If (Ψ, Θ, 0) is a PU–algebra then for every ɒ, ɗ, ɔ ∈ Ψ.

(1) (ɔ Θ ɒ) Θ (ɔ Θ ɗ) = ɒ Θ ɗ. **(2)** (ɒ Θ ɗ) Θ ɔ = (ɔ Θ ɗ) Θ ɒ.

Definition 1.6 [4] A subset J ≠ ∅ of a PU–algebra (Ψ, Θ, 0) is a PU–subalgebra of Ψ if ɒ Θ ɗ ∈ *J whenever* ɒ, ɗ ∈ *J*

Definition 1.7 [4] A subset J ≠ ∅ of a PU–algebra (Ψ, Θ, 0) is an ideal of Ψ if for every ɒ, ɗ ∈ Ψ, 0 ∈ *J* and ɒ Θ ɗ, ɒ ∈ *J* ⟹ ɗ ∈ *J*

Definition 1.8 [4] A subset J_{PK} ≠ ∅ of a PU–algebra (Ψ, Θ, 0) is a KU – ideal of Ψ if it fulfills the following (1) and (2) conditions ∀ ɒ, ɗ, ɔ ∈ Ψ.

(1) 0 ∈ J_{PK} **(2)** ɒ Θ (ɗ Θ ɔ) ∈ J_{PK} and ɗ ∈ J_{PK} imply ɒ Θ ɔ ∈ J_{PK}

Definition 1.9 [4] A subset J ≠ ∅ of a PU–algebra (Ψ, Θ, 0) is a PU_2–ideal of Ψ if ∀ ɒ, ɗ, ɔ ∈ Ψ.

(1) 0 ∈ *J* **(2)** (ɒ Θ ɗ) Θ ɔ ∈ *J*, ɔ Θ ɗ ∈ *J imply* ɒ ∈ *J*

Definition 1.10 [4] A subset J ≠ ∅ of a PU–algebra (Ψ, Θ, 0) is a PU_3–ideal of Ψ if

(1) 0 ∈ *J* **(2)** (ɒ Θ (ɗ Θ ɔ)) Θ ɔ ∈ *J*, ∀ ɒ, ɗ ∈ *J and* ɔ ∈ Ψ

Definition 1.11 [4] A subset J ≠ ∅ of a PU–algebra (Ψ, ʘ, 0) is a PU_4–ideal of Ψ if

(1) $0 \in J$ **(2)** (ɒ ʘ 0) ʘ ɗ ∈ J , ∀ ɒ, ɗ ∈ J

Definition 1.12 [4] A subset J ≠ ∅ of a PU–algebra (Ψ, ʘ, 0) is a PU- new ideal of Ψ if

(1) $0 \in J$ **(2)** (ɒ ʘ (ɗ ʘ ɔ)) ʘ ɔ ∈ J, ∀ ɒ, ɗ ∈ J *and* ɔ ∈ Ψ.

Example 1.13 Let (Ψ, ʘ, 0) is a PU–Algebra where Ψ = {0, λ, μ, ν} and ʘ is a binary operation on Ψ demarcated by the following table.

ʘ	0	λ	μ	ν
0	0	λ	μ	ν
λ	λ	0	ν	μ
μ	μ	ν	0	λ
ν	ν	μ	λ	0

Then J_1 = {0, λ}, J_2 = {0, μ}, J_3 = {0, ν} are PU-new ideals of Ψ.

Definition 1.14 [9] Let Ψ ≠ ∅ and let Ḥ be a fuzzy set in Ψ which is labeled by their membership function $Ƒ_{Ḥ}$. For any ɒ ∈ *Ψ*, this function connects a number $Ƒ_{Ḥ}$(ɒ) in the interval [0, 1]. Where the real number $Ƒ_{Ḥ}$(ɒ) is considered to be a degree of membership of ɒ ∈ Ψ to Ḥ i.e. Ḥ = {(ɒ, $Ƒ_{Ḥ}$(ɒ)) : ɒ ∈ Ψ}. A fuzzy set Ḥ in Ψ is constant if $Ƒ_{Ḥ}$ is constant.

Definition 1.15 [9] Let Ψ ≠ ∅ and let Ḥ be a fuzzy set in Ψ. The compliment of Ḥ is indicated by $\tilde{Ḥ}$ and designated by their membership function Ƒ$\tilde{Ḥ}$ which is given as follows.

Ƒ$\tilde{Ḥ}$ (ɒ) = 1 − $Ƒ_{Ḥ}$(ɒ) , ∀ ɒ ∈ Ψ

Proposition 1.16 [9] Let Ψ ≠ ∅ and let Ḥ be a fuzzy set in Ψ. Then the following statements are true, ∀ ɒ, ɗ ∈ Ψ.

(1) ($Ƒ_{Ḥ}$(ɒ) ≤ $Ƒ_{Ḥ}$(ɗ) ⇔ $Ƒ_{Ḥ}$(ɒ) ≥ $Ƒ_{Ḥ}$(ɗ)), (2) ($Ƒ_{Ḥ}$(ɒ)= $Ƒ_{Ḥ}$(ɗ) ⇔ $Ƒ_{Ḥ}$(ɒ) = $Ƒ_{Ḥ}$(ɗ))

(3) $\bar{\bar{Ḥ}}$ = Ḥ (4) (1 − min{ $Ƒ_{Ḥ}$(ɒ), $Ƒ_{Ḥ}$(ɗ)} = max {Ƒ$\tilde{Ḥ}$ (ɒ), Ƒ$\tilde{Ḥ}$ (ɗ))

Proposition 1.17 [16] Let Ψ ≠ ∅ and let $\{Ḥ_i\}_{i\in J}$ be a family fuzzy sets in Ψ while J is an indexing set. Then the statements from (1) to (9) are true:

(1) $(\operatorname{Inf}_{i\in J} \{\min \{ Ƒ_{Ḥi} (ɒ), Ƒ_{Ḥi} (ɗ)\}\} = \min \{\inf_{i\in J} \{ Ƒ_{Ḥi} (ɒ)\}, \inf_{i\in J} \{ Ƒ_{Ḥi} (ɗ)\}\})$, ∀ ɒ, ɗ ∈ Ψ

(2) $(\operatorname{Sup}_{i\in J} \{\max \{ Ƒ_{Ḥi} (ɒ), Ƒ_{Ḥi} (ɗ)\}\} = \max \{\sup_{i\in J} \{ Ƒ_{Ḥi} (ɒ)\}, \sup_{i\in J} \{ Ƒ_{Ḥi} (ɗ)\}\})$, ∀ ɒ, ɗ ∈ Ψ

(3) $(\operatorname{Inf}_{i\in J} \{\max \{ Ƒ_{Ḥi} (ɒ), Ƒ_{Ḥi} (ɗ)\}\} \geq \max \{\inf_{i\in J} \{ Ƒ_{Ḥi} (ɒ)\}, \inf_{i\in J} \{Ƒ_{Ḥi} (ɗ)\}\})$, ∀ ɒ, ɗ ∈ Ψ

(4) $(\operatorname{Sup}_{i\in J} \{\min \{Ƒ_{Ḥi} (ɒ), Ƒ_{Ḥi} (ɗ)\}\} \leq \min \{\sup_{i\in J} \{ Ƒ_{Ḥi} (ɒ)\}, \sup_{i\in J} \{ Ƒ_{Ḥi} (ɗ)\}\})$, ∀ ɒ, ɗ ∈ Ψ

(5) $(\sup_{i\in J} \{ F_{H_i}(\mathfrak{v})\})^2 = \sup_{i\in J} \{ F_{H_i}(\mathfrak{v})^2\})$, $\forall\, \mathfrak{v}, ɗ \in \Psi$

(6) $(\inf_{i\in J} \{F_{H_i}(\mathfrak{v})\})^2 = \inf_{i\in J} \{F_{H_i}(\mathfrak{v})^2\})$, $\forall\, \mathfrak{v}, ɗ \in \Psi$

(7) $(1 - \sup_{i\in J} \{F_{H_i}(\mathfrak{v})\} = \inf_{i\in J} \{1 - F_{H_i}(\mathfrak{v})\})$, $\forall\, \mathfrak{v}, ɗ \in \Psi$

(8) $(1 - \inf_{i\in J} \{ F_{H_i}(\mathfrak{v})\} = \sup_{i\in J} \{1 - F_{H_i}(\mathfrak{v})\})$. $\forall\, \mathfrak{v}, ɗ \in \Psi$

(9) For a fuzzy set F in a PU-algebra (Ψ, Θ, 0) which fulfills the following assertion:

$(F_H(\mathfrak{v}\, \Theta\, ɗ) \geq F_H(ɗ)), \forall\, \mathfrak{v}, ɗ \in \Psi \Rightarrow (F_{H(0)} \geq F_H(\mathfrak{v})), \forall\, \mathfrak{v} \in \Psi$

Let $\mathfrak{v} \in \Psi$. From (1) and (9), we get $F_{H(0)} = F_H(\mathfrak{v}\, \Theta\, \mathfrak{v}) \geq F_H(\mathfrak{v})$. $\forall\, \mathfrak{v}, ɗ \in \Psi$

Definition 1.18 [17] A fuzzy subset 'α' in Ψ where Ψ is a PU-algebra is said to be a fuzzy sub-algebra of Ψ if $\alpha(\mathfrak{v}\, \Theta\, ɗ) \geq \min\{\alpha(\mathfrak{v}), \alpha(ɗ)\}$, $\forall\, \mathfrak{v}, ɗ \in \Psi$.

Definition 1.19 [17] Let (Ψ, Θ, 0) is a PU-algebra and 'α' is a fuzzy subset in Ψ is a fuzzy new-ideal of Ψ if it fulfills **(F_1)** and **(F_2)** conditions:

(F_1) $\alpha(0) \geq \alpha(\mathfrak{v})$.

(F_2) $\alpha((\mathfrak{v}\, \Theta\, (ɗ\, \Theta\, \mathfrak{ɔ}))\, \Theta\, \mathfrak{ɔ}) \geq \min\{\alpha(\mathfrak{v}), \alpha(ɗ)\}$, $\forall\, \mathfrak{v}, ɗ, \mathfrak{ɔ} \in \Psi$.

Corollary 1.20 [17] Let 'α' is a fuzzy PU-new ideal of Ψ and if the $\mathfrak{v}\, \Theta\, ɗ \leq \mathfrak{ɔ}$ holds in Ψ, then $\alpha(ɗ) \geq \min\{\alpha(\mathfrak{v}), \alpha(\mathfrak{ɔ})\}$.

Corollary 1.21 [17] For a fuzzy subset 'α' in PU-algebra Ψ and if $\mathfrak{v} \leq ɗ$, then $\alpha(\mathfrak{v}) = \alpha(ɗ)$.

Corollary 1.22 [17] Let (Ψ, Θ, 0) and (Υ, Θ, 0) are two PU-algebras, and suppose that a function $F: \Psi \to \Upsilon$ is a homomorphism, 'α' and 'β' are the fuzzy subsets of Ψ and Υ respectively defined by

$\beta(ɗ) = \{\sup \mathfrak{v} \in F^{-1}(ɗ)\, \alpha(\mathfrak{v})$ if $F^{-1}(ɗ) = \phi$, 0 otherwise, and $\alpha(\mathfrak{v}) = \beta_{(F(\mathfrak{v}))}\}$. $\forall\, \mathfrak{v} \in \Psi$.

If 'α' is a fuzzified PU-new ideal of Ψ, then 'β' is also a fuzzified PU-new ideal of Υ.

2. PYTHAGOREAN FUZZY SETS IN PU-ALGEBRAS

The idea of Pythagorean fuzzy sets was initially suggested in 2013 by Yager [11] and Yager and Abasov [18].

Definition 2.1: Let $\Psi \neq \emptyset$, then a set 'p' is a Pythagorean fuzzy set in Ψ is designated by their α_p (membership function) and β_p (non-membership function). To each point $\mathfrak{v} \in \Psi$, α_p and β_p link the real numbers $\alpha_P(\mathfrak{v})$ and $\beta_P(\mathfrak{v})$ in [0, 1], with the condition given as follows

$(0 \leq \alpha_P(\mathfrak{v})^2 + \beta_P(\mathfrak{v})^2 \leq 1)$, $\forall\, \mathfrak{v} \in \Psi$

The values $\alpha_{ƥ}(ɒ)$ and $\beta_{ƥ}(ɒ)$ are presumed for a point 'ɒ' as a degree of membership and a degree non-membership respectively to the set 'ƥ' i.e.
$ƥ = \{(ɒ, \alpha_{ƥ}(ɒ), \beta_{ƥ}(ɒ)) \mid \forall\, ɒ \in \Psi\}$. In calculation we simply use $ƥ = (\alpha_{P}, \beta_{P})$ instead of $ƥ = \{(ɒ, \alpha_{ƥ}(ɒ), \beta_{ƥ}(ɒ)) \mid \forall\, ɒ \in \Psi\}$.

Definition 2.2 For a constant Pythagorean fuzzy set 'ƥ' in Ψ their $\alpha_{ƥ}$ (membership function) and $\beta_{ƥ}$ (non-membership function) are constant.

We smear the notion of Pythagorean fuzzy set to PU-algebras and present some new properties in PU-algebra.

Definition 2.3 A set $ƥ = (\alpha_{ƥ}, \beta_{ƥ})$ in a PU-algebra $(\Psi, \Theta, 0)$ is a Pythagorean fuzzy PU-subalgebra of Ψ if it satisfies the following statements:

$(\alpha_{P}(ɒ\ \Theta\ ɗ) \geq \min\{\alpha_{ƥ}(ɒ), \alpha_{ƥ}(ɗ)\}), \ \forall\, ɒ, ɗ \in \Psi$

$(\beta_{ƥ}(ɒ\ \Theta\ ɗ) \leq \max\{\beta_{ƥ}(ɒ), \beta_{ƥ}(ɗ)\}), \ \forall\, ɒ, ɗ \in \Psi$

Definition 2.4 Let J_{id} is an ideal of PU – algebra then its Pythagorean fuzzification is defined as

(i) $\alpha_{ƥ}(0) \geq \alpha_{ƥ}(ɒ), \quad \forall\, ɒ \in J$

(ii) $\beta_{ƥ}(0) \leq \beta_{ƥ}(ɒ), \quad \forall\, ɒ \in J$

(iii) $\alpha_{ƥ}(ɗ) \geq \min\{\alpha_{ƥ}(ɒ\ \Theta\ ɗ); \alpha_{ƥ}(ɒ)\}, \forall\, ɒ, ɗ \in \Psi$

(iv) $\beta_{ƥ}(ɗ) \leq \max\{\beta_{ƥ}(ɒ\ \Theta\ ɗ); \beta_{ƥ}(ɒ)\}, \forall\, ɒ, ɗ \in \Psi$

Definition 2.5 Let J_{ni} is a PU – new ideal of PU – algebra. Then its Pythagorean fuzzification is defined as $\forall\, ɒ, ɗ, ɔ \in \Psi$.

(i) $\alpha_{ƥ}(0) \geq \alpha_{ƥ}(ɒ)$

(ii) $\beta_{ƥ}(0) \leq \beta_{ƥ}(ɒ)$

(iii) $\alpha_{ƥ}((ɒ\ \Theta\ (ɗ\ \Theta\ ɔ))\ \Theta\ ɔ) \geq \min\{\alpha_{ƥ}(ɒ), \alpha_{ƥ}(ɗ)\}$

(iv) $\beta_{ƥ}((ɒ\ \Theta\ (ɗ\ \Theta\ ɔ))\ \Theta\ ɔ) \leq \max\{\beta_{ƥ}(ɒ), \beta_{ƥ}(ɗ)\}$

Theorem 2.6 A set $ƥ = (\alpha_{ƥ}, \beta_{ƥ})$ in a PU-algebra $(\Psi, \Theta, 0)$ is a Pythagorean fuzzy ideal in PU–algebra, if $\alpha_{ƥ}(ɒ)$ is constant and $\beta_{ƥ}(ɒ)$ is identity $\forall\, ɒ \in \Psi$.

Proof. Since the set $ƥ = (\alpha_{ƥ}, \beta_{ƥ})$ is a Pythagorean fuzzy ideal in PU–algebra. Then it satisfies (Definition 2.4 (i) and (ii)) $\forall\, ɒ \in \Psi$.

$\alpha_{ƥ}(ɒ) \geq \min\{\alpha_{ƥ}(ɒ\ \Theta\ ɒ), \alpha_{ƥ}(ɒ)\} = \min\{\alpha_{ƥ}(0); \alpha_{ƥ}(ɒ)\} = \alpha_{ƥ}(0)$ by (preposition 1.4 (a))

$\alpha_{ƥ}(ɒ) \geq \alpha_{ƥ}(0)$ (1)

$\alpha_{ƥ}(ɒ\ \Theta\ ɒ) \geq \min\{\alpha_{ƥ}(ɒ); \alpha_{ƥ}(ɒ)\}$,

By (preposition 1.4 (a)) we have $\alpha_{ƥ}(0) \geq \min\{\alpha_{ƥ}(ɒ); \alpha_{ƥ}(ɒ)\} \Longrightarrow \alpha_{ƥ}(0) \geq \alpha_{ƥ}(ɒ)$(2)

From (1) and (2) we have $\alpha_{ƥ}(0) = \alpha_{ƥ}(ɒ)$

Next we consider that by (preposition 1.4 (a)) we have $ɒ \leq ɒ$

We know that $\beta_{ƥ}(ɒ\ \Theta\ y) \leq \max\{\beta_{ƥ}(ɒ), \beta_{ƥ}(y)\}$

Replacing y by 'ɒ' in above expression we get

$\beta_{ƥ}(ɒ\ \Theta\ ɒ) \leq \max\{\beta_{ƥ}(ɒ), \beta_{ƥ}(ɒ)\}$, by (preposition 1.4 (a)) we have $\beta_{ƥ}(0) \leq \max\{\beta_{ƥ}(ɒ), \beta_{ƥ}(ɒ)\}$

$\Longrightarrow \beta_{ƥ}(0) \leq \beta_{ƥ}(ɒ)$. Similarly by (Definition 1.1) $\beta_{ƥ}(ɒ) = \beta_{ƥ}(0\ \Theta\ ɒ) \leq \max\{\beta_{ƥ}(0), \beta_{ƥ}(ɒ)\}$

$\Longrightarrow \beta_{ƥ}(ɒ) \leq \beta_{ƥ}(ɒ) \ \Longrightarrow \beta_{ƥ}(ɒ) = \beta_{ƥ}(ɒ)$

Hence the proof.

Theorem 2.7 A set $p = (\alpha_P, \beta_P)$ in a PU-algebra $(\Psi, \Theta, 0)$ is a Pythagorean fuzzy new ideal of Ψ, if the inequity $\text{ɒ} \Theta \text{ɗ} \leq \text{ɔ}$ is valid in Ψ, then $\alpha_P(\text{ɗ}) \geq \min\{\alpha_P(\text{ɒ}), \alpha_P(\text{ɔ})\}$ and $\beta_P(\text{ɗ}) \leq \max\{\beta_P(\text{ɒ}), \beta_P(\text{ɗ})\}$.

Proof. We suppose that the inequity $\text{ɒ} \Theta \text{ɗ} \leq \text{ɔ}$ is valid in Ψ. Then $\text{ɔ} \Theta (\text{ɒ} \Theta \text{ɗ}) = 0$ and by (definition (2.5)). $\alpha_P((\text{ɔ} \Theta (\text{ɒ} \Theta \text{ɗ})) * \text{ɗ}) \geq \min\{\alpha_P(\text{ɒ}), \alpha_P(\text{ɔ})\}$

$\Rightarrow \alpha_P(0 * \text{ɗ}) \geq \min\{\alpha_P(\text{ɒ}), \alpha_P(\text{ɔ}) \Rightarrow \alpha_P(\text{ɗ}) \geq \min\{\alpha_P(\text{ɒ}), \alpha_P(\text{ɔ})\}$.

Similarly $\beta_P(\text{ɗ}) \leq \max\{\beta_P(\text{ɒ}), \beta_P(\text{ɗ})\}$.

Proposition 2.8 If A set $p = (\alpha_P, \beta_P)$ in a PU-algebra $(\Psi, \Theta, 0)$ is a Pythagorean fuzzy set satisfying the following statements:

1. $\text{ɔ} \leq \text{ɒ} \Rightarrow \alpha_P(\text{ɒ} \Theta \text{ɗ}) \geq \min\{\alpha_P(\text{ɔ}), \alpha_P(\text{ɗ})\} \quad \forall\, \text{ɒ}, \text{ɗ}, \text{ɔ} \in \Psi$
2. $\text{ɔ} \leq \text{ɒ} \Rightarrow \beta_P(\text{ɒ} \Theta \text{ɗ}) \leq \max\{\beta_P(\text{ɔ}), \beta_P(\text{ɗ})\} \quad \forall\, \text{ɒ}, \text{ɗ}, \text{ɔ} \in \Psi$

Then 'p' is a Pythagorean fuzzy PU–subalgebra of Ψ.

Proof. Suppose that $\text{ɒ}, \text{ɗ} \in \Psi$ then by (proposition 1.4 (a)) we have $\text{ɒ} \leq \text{ɒ}$, it comes from (1) and (2) that $\alpha_P(\text{ɒ} \Theta \text{ɗ}) \geq \min\{\alpha_P(\text{ɒ}), \alpha_P(\text{ɗ})\}$ and $\beta_P(\text{ɒ} \Theta \text{ɗ}) \leq \max\{\beta_P(\text{ɒ}), \beta_P(\text{ɗ})\}$ respectively. Which is a (Definition 2.3). Hence 'p' is Pythagorean fuzzified PU–subalgebra of Ψ.

Theorem 2.9 If $P = (\alpha_P, \beta_P)$ is a Pythagorean fuzzy set in Ψ fulfilling the assertions:

1. $\text{ɔ} \leq \text{ɒ} \Rightarrow \alpha_P(\text{ɒ} \Theta \text{ɗ}) \geq \min\{\alpha_P(\text{ɔ}), \alpha_P(\text{ɗ})\}, \quad \forall\, \text{ɒ}, \text{ɗ}, \text{ɔ} \in \Psi$
2. $\text{ɔ} \leq \text{ɒ} \Rightarrow \beta_P(\text{ɒ} \Theta \text{ɗ}) \leq \max\{\beta_P(\text{ɔ}), \beta_P(\text{ɗ})\}, \quad \forall\, \text{ɒ}, \text{ɗ}, \text{ɔ} \in \Psi$

Then it fulfills the assertions.

3. $(\alpha_P(0) \geq \alpha_P(\text{ɒ})), \quad \forall\, \text{ɒ} \in \Psi$
4. $(\beta_P(0) \leq \beta_P(\text{ɒ})), \quad \forall\, \text{ɒ} \in \Psi$

Proof. Let $\text{ɒ}, \text{ɗ} \in \Psi$ then by (proposition 1.4 (a)) we have $\text{ɒ} \leq \text{ɒ}$. From (theorem 2.9 (1)) we get $\alpha_P(\text{ɒ} \Theta \text{ɗ}) \geq \min\{\alpha_P(\text{ɒ}), \alpha_P(\text{ɗ})\}, \quad \forall\, \text{ɒ}, \text{ɗ} \in \Psi$

If we put $\text{ɗ} = \text{ɒ}$ in the above expression we get

$\alpha_P(\text{ɒ} \Theta \text{ɒ}) \geq \min\{\alpha_P(\text{ɒ}), \alpha_P(\text{ɒ})\}, \quad \forall\, \text{ɒ} \in \Psi$

By (proposition 1.4 (a)) we have $(\alpha_P(0) \geq \alpha_P(\text{ɒ})) \quad \forall\, \text{ɒ} \in \Psi$

Next we consider that by (proposition 1.4 (a)) we have

$\text{ɒ} \leq \text{ɒ} \Rightarrow \beta_P(\text{ɒ} \Theta \text{ɗ}) \leq \max\{\beta_P(\text{ɒ}), \beta_P(\text{ɗ})\}, \quad \forall\, \text{ɒ}, \text{ɗ} \in \Psi$

If we put $\text{ɗ} = \text{ɒ}$ in the above expression then we get

$\beta_P(\text{ɒ} \Theta \text{ɒ}) \leq \max\{\beta_P(\text{ɒ}), \beta_P(\text{ɒ})\}, \quad \forall\, \text{ɒ} \in \Psi$

(By proposition 1.4 (a)) we have $\beta_P(0) \leq \beta_P(\text{ɒ}), \quad \forall\, \text{ɒ} \in \Psi$

Proposition 2.10 If $P = (\alpha_P, \beta_P)$ is a Pythagorean fuzzy set in Ψ fulfilling the following assertions.

1. $\alpha_P(\text{ɒ} \Theta \text{ɗ}) \geq \min\{\alpha_P(\text{ɔ}), \alpha_P(\text{ɗ})\}, \quad \forall\, \text{ɒ}, \text{ɗ}, \text{ɔ} \in \Psi$
2. $\beta_P(\text{ɒ} \Theta \text{ɗ}) \leq \max\{\beta_P(\text{ɔ}), \beta_P(\text{ɗ})\}, \quad \forall\, \text{ɒ}, \text{ɗ}, \text{ɔ} \in \Psi$

then it fulfills the assertions

3. $\text{ɔ} \leq \text{ɒ} \Rightarrow \alpha_P(\text{ɒ} \Theta \text{ɗ}) \geq \min\{\alpha_P(\text{ɔ}), \alpha_P(\text{ɗ})\}, \quad \forall\, \text{ɒ}, \text{ɗ}, \text{ɔ} \in \Psi$
4. $\text{ɔ} \leq \text{ɒ} \Rightarrow \beta_P(\text{ɒ} \Theta \text{ɗ}) \leq \max\{\beta_P(\text{ɔ}), \beta_P(\text{ɗ})\}, \quad \forall\, \text{ɒ}, \text{ɗ}, \text{ɔ} \in \Psi$

Proof. Proof is straight forward

Definition 2.11 Let P = (α_{Pi}, β_{Pi}) where i ∈ J is a Pythagorean fuzzy new ideal of a PU – algebra Ψ and let ɒ be an element of Ψ. We define $(\bigcap_{i\in J}\alpha Pi)$ (ɒ) = $\inf(\alpha Pi\,(ɒ))_{i\in J}$ and $(\bigcap_{i\in J}\beta Pi)(ɒ) = \inf(\beta Pi\,(ɒ))_{i\in J}$.

Theorem 2.12 The intersection of any set of Pythagorean fuzzy new ideals of a PU – algebra Ψ is also a Pythagorean fuzzy new – ideal of Ψ.

Proof. Let $\{Pi\}_{i\in J}$ be a family of Pythagorean fuzzy new – ideals of a PU – algebra Ψ. where P = (α_P, β_P) then ∀ ɒ, ɗ, ɔ ∈ Ψ.

$(\bigcap_{i\in J}\alpha Pi)(0) = \inf(\alpha Pi\,(0))_{i\in J} \geq \inf(\alpha Pi\,(ɒ))_{i\in J} = (\bigcap_{i\in J}\alpha Pi)(ɒ)$ and

$(\bigcap_{i\in J}\beta Pi)(0) = \inf\big(\beta Pi\,(0)\big)_{i\in J} \geq \inf\big(\beta Pi\,(ɒ)\big)_{i\in J} = (\bigcap_{i\in J}\beta Pi)(ɒ)$

Also $(\bigcap_{i\in J}\alpha Pi)((ɒ\,\Theta\,(ɗ\,\Theta\,ɔ))\,\Theta\,ɔ) = \inf(\alpha Pi\,(ɒ \odot (ɗ \odot ɔ))) \odot ɔ))_{i\in J}$

$\geq \inf(\min(\alpha Pi\,(ɒ), \alpha Pi\,(ɗ))_{i\in J}$

$= \min\{\inf(\alpha Pi\,(ɒ))_{i\in J},\ \inf(\alpha Pi\,(ɗ))_{i\in J}\}$

$= \min\{(\bigcap_{i\in J}\alpha Pi)(ɒ), (\bigcap_{i\in J}\alpha Pi)(ɗ)\}$

Similarly

$(\bigcap_{i\in J}\beta Pi)((ɒ\,\Theta\,(ɗ\,\Theta\,ɔ))\,\Theta\,ɔ) \leq \max\{(\bigcap_{i\in J}\beta Pi)(ɒ), (\bigcap_{i\in J}\beta Pi)(ɗ)\}$

Proposition 2.13 If P = (α_P, β_P) is a Pythagorean fuzzy KU – ideal of PU – algebra Ψ then.

(1) $a \leq ɒ\,\Theta\,(ɗ\,\Theta\,ɔ) \Rightarrow \alpha_P(a\,\Theta\,ɔ) \geq \min\{\alpha_P(ɒ), \alpha_P(ɗ)\}$, ∀ a, ɒ, ɗ, ɔ ∈ Ψ

(2) $a \leq ɒ\,\Theta\,(ɗ\,\Theta\,ɔ) \Rightarrow \beta_P(a\,\Theta\,ɔ) \leq \max\{\beta_P(ɒ), \beta_P(ɗ)\}$, ∀ a, ɒ, ɗ, ɔ ∈ Ψ

Proof. Let a, ɒ, ɗ, ɔ ∈ Ψ such that $a \leq ɒ\,\Theta\,(ɗ\,\Theta\,ɔ)$. Then $a\,\Theta\,(ɒ\,\Theta\,(ɗ\,\Theta\,ɔ)) = 0$.

So $\alpha_P(a\,\Theta\,(ɗ\,\Theta\,ɔ)) \geq \min\{\alpha_P(a\,\Theta\,(ɒ\,\Theta\,(ɗ\,\Theta\,ɔ))), \alpha_P(ɒ)\} = \min\{\alpha_P(0), \alpha_P(ɒ)\} = \alpha_P(ɒ)$ and $\beta_P(a\,\Theta\,(ɗ\,\Theta\,ɔ)) \leq \max\{\beta_P(a\,\Theta\,(ɒ\,\Theta\,(ɗ\,\Theta\,ɔ))), \beta_P(ɒ)\} = \max\{\beta_P(0), \beta_P(ɒ)\} = \beta_P(ɒ)$.

Thus $\alpha_P(a\,\Theta\,ɔ) \geq \min\{\alpha_P(a\,\Theta\,(ɗ\,\Theta\,ɔ)), \alpha_P(ɗ)\} \geq \min\{\alpha_P(ɒ), \alpha_P(ɗ)\}$

And $\beta_P(a\,\Theta\,ɔ) \leq \max\{\beta_P(a\,\Theta\,(ɗ\,\Theta\,ɔ)), \beta_P(ɗ)\} \leq \max\{\beta_P(ɒ), \beta_P(ɗ)\}$.

Hence, P fulfills the assertions (1) and (2).

Theorem 2.14 If every Pythagorean fuzzy subalgebra J of PU – algebra Ψ satisfies the conditions $\alpha_P(ɗ\,\Theta\,ɒ) \geq \alpha_P(y)$ and $\beta_P(ɗ\,\Theta\,ɒ) \leq \beta_P(y)$ then α_P and β_P are constant functions.

Proof. Since $\alpha_P(ɗ\,\Theta\,ɒ) \geq \alpha_P(ɗ)$ (3)

And $\beta_P(ɗ\,\Theta\,ɒ) \leq \beta_P(ɗ)$ (4)

Putting y = 0 in (3) we get $\alpha_P(0\,\Theta\,ɒ) \geq \alpha_P(0)$

$\Longrightarrow \alpha_P(ɒ) \geq \alpha_P(0)$ (5)

Similarly by putting y = 0 in (4) we get $\beta_P(0\,\Theta\,ɒ) \leq \beta_P(0)$

$\Longrightarrow \beta_P(ɒ) \leq \beta_P(0)$ (6)

From (theorem 2.9) we have

$\alpha_P(0) \geq \alpha_P(ɒ)$ (7)

$\beta_P(0) \leq \beta_P(ɒ)$ (8)

From (5) and (7) we get $\alpha_P(ɒ) = \alpha_P(0)$ and from (6) and (8) we get $\beta_P(ɒ) = \beta_P(0)$

Hence α_P and β_P are constant functions.

3. CONCLUSIONS

We introduced the concept of Pythagorean fuzzification in P U-algebras and investigated some of its useful properties in this paper. We believe that these results will be very useful in the development of algebraic structures. Furthermore, these definitions and main results can be similarly extended to other algebraic structures such as P S-algebras, Q-algebras, SU-algebras, IS-algebras, algebras, and semirings. We hope that this work will lay the groundwork for further research into the theory of BCI-algebras.

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