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Integral tranforms Technique for Solving linear Stochastic Differential Equation

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Abstract

In this article we have used the Laplace and Natural transform as two different integral transforms to solve a second order linear Stochastic differential equations in order to get unique solution in order to established a technique to obtain a solution of Stochastic differential equations by using integral transforms. These results will helps to improve the literature on stochastic differential equations and will be useful in applied and financial mathematics.

Keywords: Integral transforms, Natural Transform, Laplace transform , Stochastic differential Equations, 2nd order linear differential equation.

1. INTRODUCTION

Stochastic differential equations (SDE) are such equations where stochastic processes represent in one or more terms and as a consequence, the resultant solution will also be stochastic. Many researchers and mathematicians are working on SDE and discovers numerous fruitful applications in the area of physics, applied mathematics, engineering and in financial mathematical (Oksendal, 2013) there are integral transforms which are broadly used in the same fields of areas as well. The integral transforms transform such as Mellin and Hankel transform, Laplace transform, Elzaki transform, Fourier transform, Sumudu transform, Aboodh transform and ZZ transform has been remained very useful in solving ordinary, partial and fractional differential equations (Elzaki, 2011), (Aboodh,2013), (Ahmed, 2015). Few researchers successfully applied integral transforms to solve SDE, in 2018, Bright Osu and Vivian Sampson used Aboodh transform to solve SDE (Osu,2018) In 2016, ZZ transform was introduced and

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successfully applied on Differential equations by Zain-ul-Abadin Zafar (Zafar, 2016). It will very interesting to investigate to see whether all integral transforms are also effective to solve the SDE and provide unique solution if same SDE solved by different integral transforms. In this article we solved a same second order linear stochastic differential equations by two different transforms such as Laplace and Natural transforms to get a unique solution.

1.1 Laplace transform

Laplace transform is a type of integral transform introduced by a French mathematician Pierre Simon Laplace which is widely used in solving differential equations.

Laplace transform convert the differential equation from the time domain $f(t)$ into frequency domain $F(s)$. The reason of the importance of Laplace transform is that it converts ordinary differential equation into algebraic equation which is easy to solve(Vaithyasubramanian, Vinil, & Joseph, 2017). The Laplace transform has applications throughout probability theory, including first passage times of stochastic processes such as Markov chains, and renewal theory (Pavel,2019). Let $f(t)$ be a function of time t for all $t \geq 0$ then the Laplace transform of $f(t)$ is $F(s)$ is defined as:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt \qquad (1)$$

Provided that the integral exists.

1.2 Natural transform

Natural transform is introduced by Zafar Hayat Khan in 2008 (Zafar & Waqar, 2008). The roots of Natural transform is matched with Fourier transform(Fethi & Silambarasan, 2012)

The Natural transform usually deals with continuous and continuously differentiable function (Maryam & Adem, 2016). Natural transform is used for solving differential equations with real application in applied Physics, Science, and Engineering(Shehu & Weidong, 2020).

Suppose $f(t)$ be a function defined for all $t \geq 0$, then Natural transform of $f(t)$ is the function $R(u, s)$ is defined by

$$R(u, s) = N(f) = \int_0^\infty f(ut)\, e^{-st} dt \qquad (2)$$

Provided the right side integral exists.

2 RESULTS AND DISCUSSIONS

In this section we are presenting the kernel of fractional Sumudu transform and its relationship with fractional Fourier transform, which can play a significant role in solving complicated problems arising in signal processing and other fields of applied mathematics and will also helpful to obtained the solutions of fractional differential equation.

2.1 Formulation and solution of 2nd order linear Black Scholes's differential equation by Laplace and Natural transforms.

SDE is an extension of ODE allowing randomness in the system, so the solution will also have randomness. On that account, the most common class of SDE is taken, i.e.,

$$\frac{d\mathrm{x}(\mathrm{t})}{d\mathrm{t}} = \mathrm{G}(\mathrm{x}(\mathrm{t})) + \sum_{\mathrm{q}=1}^{m} \mathrm{f}_{\mathrm{q}}(\mathrm{x}(\mathrm{t}))\delta^{\mathrm{q}} \qquad (3)$$

where $\mathrm{x} \in \mathrm{X}$, and X is the space, x represents the position of the system and supposed x is differentiable manifold.

2.1.1 Formulation:

In a finite time horizon $\mathrm{T} > 0$, consider a complete probability space (Ω G P) with standard Brownian motion $\mathrm{W} = \{\mathrm{W}_{\mathrm{t}}^{1}, \mathrm{W}_{\mathrm{t}}^{2}, \dots\dots\dots \mathrm{W}_{\mathrm{t}}^{h}\}$, $0 \le \mathrm{t} \le \mathrm{T}$, valued in r^{h}, and initiating the p-augmented filtration. Non risky assert $\mathrm{S}^{0}=0$ is involved in financial market with $\mathrm{S}_{0} = 1$, means normalized to unity. Also h risky assets with price process $\mathrm{S} = (\mathrm{S}_{\mathrm{t}}^{1}, \mathrm{S}_{\mathrm{t}}^{2}, \dots\dots, \mathrm{S}_{\mathrm{t}}^{\mathrm{h}})$ whose action is defined by a SDE

$$d\mathrm{S}_{\mathrm{t}} = \mathrm{q}\mathrm{S}_{\mathrm{t}}d\mathrm{t} + \sigma\mathrm{S}_{\mathrm{t}}d\mathrm{W}_{\mathrm{t}} \qquad (4)$$

The solution of above equation (2) easily done by Itȍ formula initializing with S_{0} at $\mathrm{t} = 0$, and that is:

$$\mathrm{S}_{0} = \mathrm{S}_{0}\exp\left\{\sigma\mathrm{W}_{\mathrm{t}} + \left(\mathrm{q} - \tfrac{1}{2}\sigma^{2}\right)\mathrm{t}\right\} \ \forall\, \mathrm{t} \in [0, \mathrm{T}] \qquad (5)$$

Let consider that trading period is small, so, no new dividend will be announced and no new assets have been purchased then the stoke price follow the process

$$d\mathrm{S}_{\mathrm{t}} = \hat{\mathrm{q}}\mathrm{S}_{\mathrm{t}}d\mathrm{t} + \sigma\mathrm{S}_{\mathrm{t}}d\mathrm{W}_{\mathrm{t}}, \qquad \hat{\mathrm{q}} = \mathrm{q} + \eta \qquad (6)$$

Where η is market price of risk. The equation which satisfies the above conditions of stoke price is one variable backward Black – Scholes partial differential equation. i.e.,

$$\frac{1}{2}\sigma^{2}\mathrm{S}_{\mathrm{t}}^{2}\frac{\partial^{2}\mathrm{v}}{\partial\mathrm{S}_{\mathrm{t}}^{2}} + \mathrm{a}\mathrm{S}_{\mathrm{t}}\frac{\partial\mathrm{v}}{\partial\mathrm{S}_{\mathrm{t}}} - \mathrm{rV} = -\frac{\partial\mathrm{v}}{\partial\mathrm{t}} \qquad (7)$$

Here $\mathrm{V} = (\mathrm{S}_{\mathrm{t}}, \mathrm{H}_{\mathrm{t}})$ is the worth of investment, S_{t} represents the stoke price at time t and H_{t} is the output of investment over a period of time t.

Suppose that V represent the rate of change of worth with respect to time t, can be written as

$$\frac{\partial V(\mathrm{S}_{\mathrm{t}},\mathrm{H}_{\mathrm{t}})}{\partial\mathrm{t}} = \mathrm{P}(\mathrm{S}_{\mathrm{t}}, \mathrm{H}_{\mathrm{t}}) - \mathrm{C}(\mathrm{S}_{\mathrm{t}}, \mathrm{H}_{\mathrm{t}}) \qquad (8)$$

Where P shows the proportion rate, C shows the consumption rate and H_{t} is total output. As the rate of is consumption is very small so, $\mathrm{C}(\mathrm{S}_{\mathrm{t}}, \mathrm{H}_{\mathrm{t}}) \to 0 \ as\ \mathrm{t} \to 0$.

So, above equation (6) becomes $\frac{\partial\mathrm{v}(\mathrm{S}_{\mathrm{t}},\mathrm{H}_{\mathrm{t}})}{\partial\mathrm{t}} = \mathrm{P}(\mathrm{S}_{\mathrm{t}}, \mathrm{S}_{\mathrm{t}})$ (9)

As we know that firm's rate of worth depends upon the investment output at time t, so,

$$\frac{\partial\mathrm{v}(\mathrm{S}_{\mathrm{t}},\mathrm{H}_{\mathrm{t}})}{\partial\mathrm{t}} = \mathrm{P}(\mathrm{H}_{\mathrm{t}}) = \mathrm{S}_{\mathrm{t}} \qquad (10)$$

for short selling period. Clearly left hand side of (8) is differentiated with respect to S , therefore

$\mathrm{V}(\mathrm{S}, \mathrm{H})$ can be written as $\mathrm{V}(\mathrm{S})$. Thus equation (5) reduces to

$$\frac{1}{2}\sigma^{2}\mathrm{S}^{2}\frac{\partial^{2}\mathrm{V}}{\partial\mathrm{S}_{\mathrm{t}}^{2}} + \mathrm{q}\mathrm{S}\frac{\partial\mathrm{V}}{\partial\mathrm{S}_{\mathrm{t}}} - \mathrm{rV} = \mathrm{P}(\mathrm{H}) = -\mathrm{S} \qquad (11)$$

where V is the price of the option as a function of stock price S and time t, r is the risk-free interest rate, and σ is the volatility of the stock.

Also known as 2nd order linear Black Scholes's differential equation

2.1.2 Solution by using Laplace transform

Consider the equation (11) with initial condition

$$\frac{1}{2}\sigma^2 S\frac{dv^2}{ds^2} + ąS\frac{dv}{ds} - rV = -S \quad (12)$$

$V(0) = b$, $V'(0) = c$, Then the solution is given as:

$$V(S) = -\frac{1}{\alpha\omega}S + \frac{\alpha^3 b - 1}{\alpha^2(\alpha - \omega)}e^{\alpha S} + \frac{\omega^3 b - 1}{\omega^2(\omega - \alpha)} \quad (13)$$

Proof:

Assume that 'm' and 'n' be the roots of the homogeneous part of (10) , then

$$m + n = \frac{-2\alpha}{S\sigma^2} = -\frac{1}{S} \quad (14)$$

$$mn = \frac{-2(S+ą)}{\sigma^2 S^2} = -\frac{S+ą}{ąS^2} \quad (15)$$

Now root's equation becomes:

$$\frac{d^2V}{dS^2} - (m+n)\frac{dV}{dS} + mn\,V = -S \quad (16)$$

Applying Laplace transform:

$$\mathcal{L}\left\{\frac{d^2V}{dS^2}\right\} - (m+n)\mathcal{L}\left\{\frac{dV}{dS}\right\} + mn\,\mathcal{L}\{V\} = \mathcal{L}\{-S\}$$

$$[q^2V(q) - qV(0) - V'(0)] - (m+n)[qV(q) - V(0)] + mnV(q) = \frac{-1}{q^2}$$

Applying initial conditions:

$$[q^2V(q) - qb - c] - (m+n)[qV(q) - b] + mnV(q) = \frac{-1}{q^2}$$

$$(q^2 - (m+n)q + mn)V(q) = \frac{-1}{q^2} + qb + c - (m+n)b$$

$$V(q) = \frac{-1 + q^3b + q^2c - q^2b(m+n)}{q^2(q^2 - (m+n)q + mn)}$$

$$V(q) = \frac{-1 + q^3b}{q^2(q^2 - (m+n)q + mn)} + \frac{q^2(c - b(m+n))}{q^2(q^2 - (m+n)q + mn)} \quad (17)$$

Resolving partial fraction, the result is:

$$V(q) = \frac{-1}{\alpha\omega q^2} + \frac{\alpha^3 b - 1}{\alpha^2(\alpha - \omega)(q - \alpha)} + \frac{\omega^3 b - 1}{\omega^2(\omega - \alpha)(q - \omega)} + \frac{(c - b(m+n))}{(\alpha - \omega)(q - \alpha)} + \frac{(c - b(m+n))}{(\omega - \alpha)(q - \omega)}$$

After applying inverse Laplace transform, result is similar as in equation

$$V(S) = -\frac{1}{\alpha\omega}S + \frac{\alpha^3 b - 1}{\alpha^2(\alpha - \omega)}e^{\alpha S} + \frac{\omega^3 b - 1}{\omega^2(\omega - \alpha)} \quad (18)$$

Where $\alpha = \frac{(m+n)+\sqrt{(m+n)^2 - 4mn}}{2}$ and $\omega = \frac{(m+n)-\sqrt{(m+n)^2 - 4mn}}{2}$

2.1.2 Solution by using Natural transform

Now applying Natural transom on equation (10) and using equations (11) to (13)

$$N\left\{\frac{d^2V}{dS^2}\right\} - (m+n)N\left\{\frac{dV}{dS}\right\} + mnN\{V\} = N\{-S\}$$

$$\left(\frac{s^2}{u^2}R(s,u) - \frac{s}{u^2}v(0) - \frac{V'(0)}{u}\right) - (m+n)\left(\frac{s}{u}R(s,u) - \frac{V(0)}{u}\right) + mn\,R(s,u) = \frac{u}{s^2}$$

$$\left(\frac{s^2}{u^2}R(s,u) - \frac{s}{u^2}.b - \frac{c}{u}\right) - (m+n)\left(\frac{s}{u}R(s,u) - \frac{b}{u}\right) + mn\,R(s,u) = -\frac{u}{s^2}$$

$$R(s,u)\left(\frac{s^2}{u^2} - (m+n)\frac{s}{u} + mn\right) = -\frac{u}{s^2} + \frac{sb}{u^2} + \frac{c}{u} - (m+n)\frac{b}{u}$$

$$= -\frac{1}{\frac{s}{u}.s} + \frac{s}{u}.\frac{b}{u} + \frac{c}{u} - (m+n)\frac{b}{u}$$

Put $q = \frac{s}{u} \Rightarrow s = uq$

Then

$$Ɍ(s,u)(q^2-(m+n)q+mn) = -\frac{1}{q.uq} + \frac{b}{u}q + \frac{c}{u} - (m+n)\frac{b}{u}$$

$$= \frac{-1+q^3b+cq^2-(m+n)bq^2}{uq^2}$$

$$= \frac{q^3b-1+q^2\,(c-(m+n)b)}{uq^2}$$

$$Ɍ(s,u) = \frac{q^3b-1}{uq^2\,(q^2-(m+n)q+mn)} + \frac{q^2\,(c-(m+n)b)}{uq^2\,(q^2-(m+n)q+mn)}$$

$$Ɍ(s,u) = \frac{1}{u}\left(\frac{q^3b-1}{q^2\,(q^2-(m+n)q+mn)}\right) + \frac{1}{u}\left(\frac{(c-(m+n)b)}{(q^2-(m+n)q+mn)}\right)$$

$$= \frac{1}{u}\left(\frac{q^3b-1}{q^2\,(q-\alpha)(q-\omega)}\right) + \frac{1}{u}\left(\frac{(c-(m+n)b)}{(q-\alpha)(q-\omega)}\right)$$

Resolving partial fractions;

$$Ɍ(s,u) = \frac{1}{u}\left(-\frac{1}{\alpha\omega q^2} + \frac{\alpha^3b-1}{\alpha^2(\alpha-\omega)(q-\alpha)} + \frac{\omega^3b-1}{\omega^2(\omega-\alpha)(q-\omega)}\right)$$

$$+\frac{1}{u}\left(\frac{(c-(m+n)b)}{(\alpha-\omega)(q-\alpha)} + \frac{(c-(m+n)b)}{(\omega-\alpha)(q-\omega)}\right) \qquad (17)$$

Substituting $q = \frac{s}{u}$ in eqn (16)

$$Ɍ(s,u) = \frac{1}{u}\left(-\frac{1}{\alpha\omega(\frac{s}{u})^2} + \frac{\alpha^3b-1}{\alpha^2(\alpha-\omega)\left(\frac{s}{u}-\alpha\right)} + \frac{\omega^3b-1}{\omega^2(\omega-\alpha)\left(\frac{s}{u}-\omega\right)}\right)$$

$$+\frac{1}{u}\left(\frac{(c-(m+n)b)}{(\alpha-\omega)\left(\frac{s}{u}-\alpha\right)} + \frac{(c-(m+n)b)}{(\omega-\alpha)\left(\frac{s}{u}-\omega\right)}\right)$$

$$= \frac{1}{u}\left(-\frac{1}{\alpha\omega}.\frac{1}{\frac{s^2}{u^2}} + \frac{\alpha^3b-1}{\alpha^2(\alpha-\omega)}.\frac{1}{\left(\frac{s}{u}-\alpha\right)} + \frac{\omega^3b-1}{\omega^2(\omega-\alpha)}.\frac{1}{\left(\frac{s}{u}-\omega\right)}\right)$$

$$+\frac{1}{u}\left(\frac{(c-(m+n)b)}{(\alpha-\omega)}.\frac{1}{\left(\frac{s}{u}-\alpha\right)} + \frac{(c-(m+n)b)}{(\omega-\alpha)}.\frac{1}{\left(\frac{s}{u}-\omega\right)}\right) \qquad (19)$$

Applying inverse natural transform on eqn (17) we get

$$V(S) = -\frac{1}{\alpha\omega}S + \frac{\alpha^3b-1}{\alpha^2(\alpha-\omega)}e^{\alpha S} + \frac{\omega^3b-1}{\omega^2(\omega-\alpha)}e^{\omega S} \qquad (20)$$

where $\alpha = \frac{(m+n)+\sqrt{(m+n)^2-4mn}}{2}$ and

$$\omega = \frac{(m+n)-\sqrt{(m+n)^2-4mn}}{2}$$

Equation (17) and (19) are same which indicates unique solution after applying two different transforms on equation (12)

CONCLUSION:

A second order linear stochastic differential equation is formulated and its unique solution obtained by using two different integral transforms namely Laplace and Natural transforms and it is found that the stochastic differential equations or Black-Scholes equation are solvable by using different integral transforms such concepts will highly useful in financial mathematics specially in stock marketing

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