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Integral transform method for solving homogeneous and inhomogeneous fractional differential equations using Laplace and Natural transforms

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Abstract:

In this article we have used the Laplace and Natural transform transforms to solve homogenous and inhomogeneous fractional differential equations in order to get a unique solution in order to establish a technique of solving fractional differential equations by using different integral transforms. These results will be helpful in various applications of engineering and different branches of physics and applied mathematics

Keywords: Natural Transform, Laplace transform, fractional differential equations, homogenous and inhomogeneous forms

1. INTRODUCTION

The idea and concept of fractional calculus (FC) was actually introduced in middle of nineteenth century and for last twenty years engineers and physicists have found many applications of FC in their areas. In particular, typical mathematical works provide extensive findings on aspects with reasonably little significance in applications, and the engineering literature frequently lacks mathematical detail and accuracy. Mathematicians worked on the area of fractional derivatives and make possible to applicable in most important applications, the Caputo operators, which provides a self-contained, detailed and mathematically difficult study of their properties and of the corresponding differential equations which is known as fractional differential equations (Diethelm, 2002)

The fractional differential equations getting the attention of physicists and mathematicians for the last twenty years and now days it is very popular among the

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mathematicians due to its numerous important applications in the field of applied mathematics and engineering, such applications are based on largely on the mathematically analysis. Some recent contributions to the theory of fractional differential equations can be seen in (Miller, et al, 1993)

FC contains the concepts and the methods for solving the differential equations involving derivatives of non-integer orders. Fractional differential equations used in many areas of studies such as optics, signal processing, quantum mechanics, chemistry, and biology. there . There are many methods to solve the FDEs, such as numerical approach (Abdon & Aydin, 2013). Adomian's decomposition method (Abbasbandy, 2007). In this article we have applied Laplace and Natural transforms method to solve homogenous and inhomogeneous fractional differential equations.

1.1 Methodology and Preliminary concepts

1.1.1 Laplace transform

Laplace transform is widely used in solving classical differential equations. Laplace transform convert the differential equation from the time domain $f(t)$ into frequency domain $F(s)$. The reason of the importance of Laplace transform is that it converts differential equations into algebraic equation (Reddy, et al, 2017) The Laplace transform is defined as if $f(t)$ be a function of time t for all $t \geq 0$ then the Laplace transform of $f(t)$ is expressed as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt \qquad (1)$$

Provided that the integral exists.

1.1.2 Natural transform

Natural transform and its properties are induced in (Khan & Khan, 2008).

The Natural transform usually deals with continuous and continuously differentiable function. Natural transform is used for solving differential equations with real application in applied Physics, Science, and Engineering(Belgacem, & Silambarasan,2012). Natural transform is defined as

If $f(t)$ be a function defined for all $t \geq 0$, then Natural transform then the Laplace transform of $f(t)$ is expressed as

$$R(u, s) = N(f) = \int_0^\infty f(ut) e^{-st} dt \qquad (2)$$

Provided the right side integral exists.

2 RESULTS AND DISCUSSIONS

2.1 Solution of homogeneous fractional differential equation.

Consider homogeneous linear fractional differential equation with initial condition.

$${}_0^C D_x^\alpha y(x) + a y(x) = 0, x > 0, \qquad y(0) = C, \qquad 0 < \alpha < 1 \qquad (3)$$

Let us applying Laplace, transforms to solve equation (3) by using concept of Caputo fractional derivative

$${}_a^C D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} \frac{d^n}{dx_n} f(t), \qquad n-1 < \alpha \leq n \quad (4)$$

Apply Laplace transform on both sides to eq(4) we get

$$Y(S)[S^\alpha + a] - S^{\alpha-1} y(0) = 0$$

Using initial condition $y(0) = 0$

we get

$$[S^{\propto} + a]Y(S) = C.S^{\propto-1}.\frac{1}{S^{\propto} + a} \qquad (5)$$

Now by applying inverse Laplace transform we get

$$\mathrm{y(x)} = C.E_{\alpha,1}(-a.x^{\propto}) \qquad (6)$$

Now solving equation (4) by using Natural transform

$$\frac{s}{u}R(s,u) - y(0)\frac{1}{u} + aR(s,u) = 0 \qquad (7)$$

Using initial condition $y(0) = C$

$$R(s,u)\left[\frac{s}{u} + a\right] = C\,\frac{1}{u} \qquad (8)$$

By applying inverse Natural transform

$$\mathrm{y(x)} = C.E_{\alpha,1}(-a.x^{\propto}) \qquad (9)$$

equations (6), (9) confirm the uniqueness solution of homogenous fractional differential equation after applying different integral transforms.

2.2 Solution of inhomogeneous fractional differential equation.

Let us consider the inhomogeneous FDE with initial condition

$${}_0^C D_x^{\alpha} y(x) + ay(x) = h(x), x > 0, \qquad y(0) = C, \qquad 0 < \alpha < 1 \qquad (10)$$

Let us discuss the solution of equation (10) by using Laplace and Natural transforms

Applying Laplace transform to eqn (10)

$$S^{\propto}Y(S) - \sum_{k=0}^{n-1}.S^{\propto-k-1}.ay^{(k)}(0) + Y(S) = H(s) \qquad (11)$$

Where $n = 1$ and using initial condition

$$Y(S)[S^{\propto} + a] - S^{\propto-1}C = H(S) \qquad (12)$$

$$[S^{\propto} + a]Y(S) = H(S) + C\,S^{\propto-1} \qquad (13)$$

$$Y(S) = CS^{\propto-1}.\frac{1}{S^{\propto} + a} + H(S).\frac{1}{S^{\propto} + a} \qquad (14)$$

By applying Laplace Inverse and using convolution theorem eq (14) becomes

$$y(x) = \int_0^x (x-\tau)^{\beta-1}E_{\beta-\alpha,2\alpha}(-ax+\tau)^{\beta-\propto}.h(\tau)d\tau \qquad (15)$$

Consider equation (10) and applying Natural transform on both sides

$$\frac{S^{\propto}}{U^{\propto}}R(S,U) - a\frac{S^{\propto-1}}{U^{\propto}}.C + \mathrm{aR(S,U)} = \mathrm{G(S,U)} \qquad (16)$$

Applying inverse natural transform on both sides and using convolution theorem we get

$$y(x) = \int_0^x (x-\tau)^{\beta-1}E_{\beta-\alpha,2\alpha}(-ax+\tau)^{\beta-\propto}.h(\tau)d\tau \qquad (17)$$

equations (15) and (17) are same which confirm the uniqueness in the solution of inhomogeneous fractional differential equation after applying different integral transforms.

CONCLUSION:

We have solved the homogenous and inhomogeneous linear fractional differential equations by using the Laplace and Natural transforms and obtained the unique solutions after applying different integral transforms these mathematical results will be helpful in various applications in the area of studies of engineering, signal processing and applied mathematics.

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