EUROPEAN ACADEMIC RESEARCH
Vol. X, Issue 9/ December 2022

Impact Factor: 3.4546 (UIF)
DRJI Value: 5.9 (B+)

ISSN 2286-4822
www.euacademic.org

Elzaki, Sumudu and Laplace transforms method for solving ordinary fractional differential equations

FOZIA TAJ
Email: fozia7204@gmail.com
SALEEM IQBAL[1]
Department of Mathematics, University of Balochistan, Quetta 87300, Pakistan
Email: saleemiqbal81@yahoo.com
FARHANA SARWAR
Department of Mathematics F.G.Girls Degree College
Madrissa Road, Quetta, Cantt, 87300, Pakistan
Email: f_saleem10@yahoo.com
ABDUL REHMAN
Department of Mathematics, University of Balochistan, Quetta 87300, Pakistan
Email: rehman_maths@hotmail.com

Abstract

In this article we have used the Elzaki, Sumudu and Laplace transforms to solve ordinary fractional differential equations in order to get a unique solution by applying different integral transforms. These results will be help full in establishing the concepts more clear in the way that integral transforms equally useful in solving fractional differential equations as useful for solving classical differential equations.

Keywords: Elzaki transform, Sumudu transform, Laplace transform, fractional differential Equations

1. INTRODUCTION

The integral transforms is a powerful technique in solving ordinary differential equations , partial differential equations, Stochastic differential equations, there are various types of integral transform like Laplace, Fourier, Hilbert, Hankle, Mellin and many more and most of them effectively used in various branches of physics, engineering, financial mathematics, (Cotta, 1993; Aggarwal,2019). Recently Elzaki and Sumudu transforms are getting attentions of mathematicians due to its simplest approach in solving differential equations as compare of other techniques.

Sumudu transform is introduced by Watugala in 1993 (Watugala,1993). Due to its similarity with Laplace transform it is further applied in solving ordinary differential equations and in control engineering problems. Various other properties are also discussed in (Asiru,2002 ; Eltayeb, 2010). In 2011 Elzaki introduced a new transform and named it as Elzaki transform (Elzaki,2011), soon after its introductions it attracted the researchers and mathematicians, so they worked on number of its properties and application (Elzaki,2011; Verma, 2020). In this article we have applies three different transforms namely Elzaki, Sumudu and Laplace transforms to solve

[1] Corresponding author

ordinary fractional differential equation in order to investigate the uniqueness of solution in a similar way as in (Urooj, et al, 2022). It is important to note that all integral transforms contains unknown function say $f(t)$, which is required to be determined appears under the integral sign with lower and upper limits there is a factor along with unknown function of type K(s, t) called as kernel function. Various problems arising in engineering, physics, applied mathematics and financial mathematics are expressed as modelling and leads to Partial differential equations, fractional differential equations and stochastic differential equations. There are several techniques are used in order to solve differential equations but integral technique proving very effective in solving fractional differential equations as compare to numerical methods or other methods.

1.1 Methodology and Preliminary concept

1.1.1 Elzaki transform:

Elzaki transform is derived from the classical Fourier integral based on the mathematical simplicity of the Elzaki transform and its fundamental properties. Elzaki transform was introduced to facilitate the process of solving ordinary and partial differential equations in the time domain function. The Elzaki transform denoted by the operator $E(.)$ and define by the following expression

$$E[f(t)] = T(v) = v\int_0^\infty f(t)e^{\frac{-t}{v}}dt \ \ ,t \geq 0 \qquad (1)$$

where the variable v in this transform is used to factor the variable t in the argument of the function f and $f(t)$ is the unknown function and $e^{\frac{-t}{v}}$ is the kernel of Elzaki transform (Elzaki, 2011). The Elzaki transform is very useful to solve multiple types of problems, such as initial and boundary value problems in applied sciences, engineering fields, applied mathematics, physics and aerospace sciences. This newly emerged approach has been applied in this article successfully for obtaining the solution of fractional ordinary differential equations. Elzaki transform technique is a direct, strong and valuable tool especially for the solution of fractional differential equations.

1.1.2 Sumudu transform

Soon after the introduction of Sumudu transform, it becomes very effective transform for solving many engineering and mathematical problems without resorting to a new frequency domain. This transform method is applied to get the solution of fractional ordinary differential equations. The Sumudu transform is defined by the following expression

$$G(u) = S[f(t)] = \int_0^\infty f(ut)e^{-t}dt, \ \ u \in (-t_1, t_2) \qquad (2)$$

where $G(u)$, is mentioned to known as the Sumudu transform of $f(t)$. The Sumudu transform technique can be used to solve various types problems like initial value and boundary value problems, homogeneous and non-homogeneous fractional ordinary differential equations (Watugala, 1993). In this article we have attempted to solve fractional ordinary differential equation.

1.1.3 Laplace transform

One of the oldest integral transform among the family of integral transform is Laplace transform and it is very useful tool in solving all types of differential equations mathematically it is defined

$$F(s) = \mathrm{L}[f(t)] = \lim_{t\to\infty}\int_0^t e^{-st} f(t)dt \qquad (3)$$

where $f(t)$ is the real time domain function and $F(s)$ is the complex frequency domain function (Schiff, 1999)

2. RESULTS AND DISCUSSIONS

Fractional differential equation is the generalized of a classical differential equation very useful in many areas of engineering and applied mathematics a detail theory is available in (Lakshmikantham, 2008). In this section we have presented the solution of fractional differential equation by method of integral transforms.

2.1. Elzaki transform method

Let us consider the following ordinary fractional differential equation

$$D^w f(t) + f(t) = \frac{2t^{2-w}}{\Gamma[3-w]} - \frac{t^{1-w}}{\Gamma[2-w]} + t^2 - t \qquad (4)$$

$$0 < w \le 1, \text{ with } D^{w-1}f(t)|t = 0$$

Applying Elzaki transform to eqn (4) and using initial condition

$$E[D^w f(t)] + E[f(t)] = E\left[\frac{2t^{2-w}}{\Gamma[3-w]}\right] - E[\ ] + E[t^2] - E[t\ ]$$

$$\frac{T(v)}{v^w} - \frac{D^{w-1}f(t)}{v}|\mathrm{t} = 0 + \mathrm{T(v)} = \frac{2}{\Gamma[3-w]}E[t^{2-w}] - \frac{1}{\Gamma[2-w]}E[t^{1-w}] + E[t^2] - E[t]$$

$$\frac{T(v)}{v^w} + \mathrm{T(v)} = \frac{2v^{4-w}}{\Gamma[3-w]}\Gamma[3-w] - \frac{v^{3-w}}{\Gamma[2-w]}\Gamma[2-w] + 2v^4 - v^3$$

$$\left(\frac{1}{v^w} + 1\right)\mathrm{T(v)} = 2v^{4-w} - v^{3-w} + 2v^4 - v^3$$

$$(1 + v^w)T(v) = 2v^4 - v^3 + 2v^{4+w} - v^{3+w}$$

$$(1 + v^w)T(v) = v^3(2v - 1) + v^{3+w}(2v - 1)$$

$$(1 + v^w)T(v) = (2v - 1)(v^3 + v^{3+w})$$

$$(1 + v^w)T(v) = v^3(2v - 1)(1 + v^w)$$

$$T(v) = 2v^4 - v^3 \qquad (5)$$

By taking invers Elzaki transform

$$E'[T(v)] = E'[2v^4] - E'[v^3]$$

$$f(t) = t^2 - t \qquad (6)$$

2.2. Sumudu transform method

Applying Sumudu transform to eqn (4) and using initial condition

$$S[D^w f(t)] + S[f(t)] = S\left[\frac{2t^{2-w}}{\Gamma[3-w]}\right] - S\left[\frac{t^{1-w}}{\Gamma[2-w]}\right] + S[t^2] - s[t]$$

$$\frac{F(u)}{u^w} - \frac{D^{w-1}f(t)}{u}\bigg|\mathrm{t} = 0 + F(\mathrm{u}) = \frac{2}{\Gamma[3-w]}S[t^{2-w}] - \frac{1}{\Gamma[2-w]}S[t^{1-w}] + S[t^2] - S[t]$$

$$\frac{F(u)}{u^w} + F(\mathrm{u}) = \frac{2u^{2-w}}{\Gamma[3-w]}\Gamma[3-w] - \frac{u^{1-w}}{\Gamma[2-w]}\Gamma[2-w] + 2u^2 - u$$

$$\left(\frac{1}{u^w} + 1\right)F(u) = 2u^{2-w} - u^{1-w} + 2u^2 - u$$

$$(1 + u^w)F(u) = 2u^2 - u + 2u^{2+w} - u^{1+w}$$

$$(1+u^w)\text{F(u)}=u(2u-1)+u^{1+w}(2u-1)$$
$$(1+u^w)\text{F(u)}=(2u-1)(2u+u^{1+w})$$
$$(1+u^w)\text{F(u)}=\frac{\text{u}(2u-1)(u^w+1)}{(1+u^w)}$$
$$F(u)=2u^2-u \qquad (7)$$

Now applying inverse Sumudu transform

$$S'[F(u)]=S'[2u^2-u]$$
$$f(t)=t^2-\text{t} \qquad (8)$$

2.3. Laplace transform method

Applying Sumudu transform to eqn (4) and using initial condition

$$\text{L}[\text{D}^\text{w}\text{y(t)}+\text{L}[\text{y(t)}]=\text{L}\left[\frac{2\text{t}^{2-\text{w}}}{\Gamma[3-\text{w}]}\right]-\text{L}\left[\frac{\text{t}^{1-\text{w}}}{\Gamma[2-\text{w}]}\right]+\text{L}[\text{t}^2]-\text{L}[\text{t}]$$
$$\frac{sF(s)-y(0)}{s^{1-w}}+F(S)=\frac{2}{s^{3-w}}-\frac{1}{s^{2-w}}+\frac{2}{s^3}-\frac{1}{s^2}$$

$$\frac{sF(s)s^w}{s}+F(S)=\frac{2s^w}{s^3}-\frac{s^w}{s^2}+\frac{2}{s^3}-\frac{1}{s^2}$$
$$F(s\,)(s^w+1)=(s^w+1)\left(\frac{2}{s^3}-\frac{1}{s^2}\right)$$
$$F(s\,)=\left(\frac{2}{s^3}-\frac{1}{s^2}\right) \qquad (9)$$

By applying the inverse Laplace transform

$$L^{-1}[F(s)]=L^{-1}\left(\frac{2}{s^3}-\frac{1}{s^2}\right)$$

We get $$y(t)=t^2-t \qquad (10)$$

Equations (6), (8) and (10) showing same solution so we successfully obtained the unique solution after applying these transforms.

CONCLUSION:

We have solved the ordinary linear fractional differential equations by using the Elzaki, Sumudu and Laplace tra8rms and obtained the unique solutions and established the idea that integral transform methods are equally useful to solve fractional differential equations as integral transforms are useful in solving classical differential equations.

REFERENCES

1. Aggarwal, S., & Sharma, N. (2019). Laplace transform for the solution of first kind linear Volterra integral equation. *Journal of Advanced Research in Applied Mathematics and Statistics*, *4*(3&4), 16-23.
2. Asiru, M. A. (2002). Further properties of the Sumudu transform and its applications. *International journal of mathematical education in science and technology*, *33*(3), 441-449.
3. Cotta, R. M., & Mikhailov, M. D. (1993). Integral transform method. *Applied mathematical modelling*, *17*(3), 156-161.
4. Eltayeb, H., & Kilicman, A. (2010). A note on the Sumudu transforms and differential equations. *Applied Mathematical Sciences*, *4*(22), 1089-1098.
5. Elzaki, T.M. (2011).The new integral transform "Elzaki transform." Glob. J. Pure Appl. Math. 7, 1, Elzaki, T. M. (2011). Application of new transform "Elzaki transform" to partial differential equations. *Global Journal of pure and applied Mathematics*, *7*(1), 65-70.

6. Lakshmikantham,V.,&Vatsala,A.S.(2008).Basic theory of fractional differential quations. *Nonlinear Analysis: Theory, Methods & Applications*, *69*(8), 2677-2682
7. Urooj,M., Saleem, I.,Farhana,.S, Abdul,R.(2022). Integral transform method for solving homogeneous and inhomogeneous fractional differential equations using Laplace and Natural,EUR, vol.X,issue 8, 2745-2748.
8. Schiff, J. L. (1999). *The Laplace transform: theory and applications*. Springer Science & Business Media.
9. Verma, D., & Alam, A. (2020). Analysis of simultaneous differential equations By Elzaki transform approach. *Science, Technology And Development*, *9*(1), 364-367–64.