EUROPEAN ACADEMIC RESEARCH
Vol. II, Issue 3/ June 2014

Impact Factor: 3.1 (UIF)
DRJI Value: 5.9 (B+)

ISSN 2286-4822
www.euacademic.org

Numerical Integration by Different Numerical Techniques

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Abstract:

The study investigated the effect of strategic content learning on mathematics achievement of students with learning disability. It employed a quasi experimental non-equivalent control group, pretest-posttest research design. The population was all 864 senior secondary II students in the 14 public secondary schools and a sample of all 47 (males-23, females-24) senior secondary II students drawn from intact classes in four randomly sampled coeducation secondary schools in Uzo-uwani local government area of Nsukka education zone, Enugu State, Nigeria. Two instructional programmes were developed and used for the study; they are mathematics/strategic content learning instructional plan (MSCLIP) and Mathematics conventional teaching lesson plan (MCTLP). Two research questions and two null hypotheses guided the study. The findings revealed that intervention with strategic content learning significantly improved mathematics achievement of students with mathematics learning disability and that gender as a factor does not have a significant influence on the mathematics achievement of students with mathematics learning disability taught using strategic content learning.

Key words: Strategic Content Learning, Mathematics, Achievement, Gender and Learning Disability

I. Introduction

Mathematics is the study of numbers and quantities and their relationships. Mathematics requires an understanding of the logic and rules used to solve numerical problems. Numerical analysis is an important part of mathematics known as the study of quantitative approximations to the solutions of mathematical problems including consideration of and bounds to the errors involved. Numerical analysis involves the study of methods of computing numerical data. (In many problems this implies producing a sequence of approximations by repeating the procedure again and again). People who employ numerical methods for solving problems have to worry about the following issues: the rate of convergence (how long does it take for the method to and the answer), the accuracy (or even validity) of the answer and the completeness of the response (do other solutions, in addition to the one found that exist).Numerical methods provide approximations to the problems in question .No matter how accurate they are they do not, in most cases, provide the exact answer. In some instances working out the exact answer by a deferent approach may not be possible or may be too time consuming and it is in these cases where numerical methods are most often used. In numerical analysis, **numerical integration** constitutes a broad family of algorithms for calculating the numerical value of a definite integral, and by extension, the term is also sometimes used to describe the numerical solutions of differential equations. Here we focuses on calculation of define integrals. The term **numerical quadrature** (often abbreviated to *quadrature*) is more or less a synonym for *numerical integration*, especially as applied to one-dimensional integrals. Numerical integration over more than one dimension is sometimes described as

cubature, although the meaning of *quadrature* is understood for higher dimensional integration as well. The basic problem in numerical integration is to compute an approximate solution to a definite integral.

$$\int_a^b f(x)dx$$

If *f(x)* is a smooth function integrated over a small number of dimensions, and the domain of integration is bounded, there are many methods for approximating the integral to the desired precision.

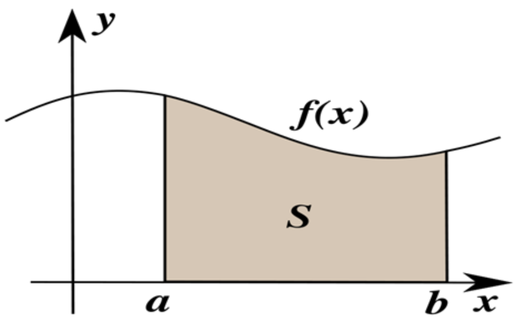


Fig (1.1) Numerical integration consists of finding numerical approximations for the value S **.**

Methodology

On the basis of available literature we have studied different methods of numerical integration: Trapezoidal, Simpson's One-Third and Simpson's Three-Eight, Gaussian Integration, Euler-McLaren Integration and Romberg Integration. Using these methods we have solved numerical problems, and done a comparative study of these methods. We have also solved nonlinear integration problem of civil engineering by using numerical integration. **Newton-cote's *Quadrature* Formula:**

Let $I = \int_a^b y dx$ where y takes the values $y_0, y_1, y_2, \ldots\ldots\ldots, y_n$ for $x = x_0, x_1, x_2, \ldots\ldots, x_n$ .Let the interval of integration (*a*, *b*) be divided into *n* equal sub-intervals, each of width $h=\frac{b-a}{n}$ so that $x_0=a, x_1=x_0+h, x_2=x_0+2h,.........., x_n=x_0+nh=b.$

$$I=\int_{x_0}^{x_0+nh} f(x)dx$$

$$I=nh\left[y_0+\frac{n}{2}\Delta y_0+\frac{n(2n-3)}{12}\Delta^2 y_0+\frac{n(n-2)^2}{24}\Delta^3 y_0+........\right] \qquad (1)$$

This is a general *quadrature* formula and is known as Newton-Cote's *quadrature* formula. A number of important deductions viz. Trapezoidal rule, Simpson's one-third and three-eighth rules, can be immediately deduced by putting n = 1, 2 and 3 respectively, in formula (1).

Trapezoidal Rule (n = 1).

Putting n = 1 in formula (1) and taking the curve through (x_0, y_0) and (x_1, y_1) as a polynomial of degree one so that differences of order higher than one vanish, we get

$$\int_{x_0}^{x_0+h} f(x)dx=h\left(y_0+\frac{1}{2}\Delta y_0\right)=\frac{h}{2}[2y_0+(y_1-y_0)]=\frac{h}{2}(y_0+y_1)$$

Similarly, for the next sub-interval $(x_0+h,\ x_0+2h)$, we get

$$\int_{x_0+h}^{x_0+2h} f(x)dx=\frac{h}{2}(y_1+y_2),............,\quad \int_{x_0+(n-1)h}^{x_0+nh} f(x)dx=\frac{h}{2}(y_{n-1}+y_n)$$

Adding the above integrals, we get

$$\int_{x_0}^{x_0+nh} f(x)dx=\frac{h}{2}[(y_0+y_n)+2(y_1+y_2+..........+y_{n-1})]$$

This is known as Trapezoidal rule. By increasing the number of subintervals, thereby making h very small, we can improve the accuracy of the value of the given integral.

Simpson's One-Third Rule (n = 2).

Putting n = 2 in formula (1) and taking the curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) as a polynomial of degree two so that differences of order higher than two vanish, we get,

$$\int_{x_0}^{x_0+2h} f(x)dx = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$

$$= \frac{2h}{6}[6y_0 + 6(y_1 - y_0) + (y_2 - 2y_1 + y_0)] = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly, $\int_{x_0+2h}^{x_0+4h} f(x)dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$,..........,

$$\int_{x_0+n(n-2)h}^{x_0+nh} f(x)dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

Adding the above integrals, we get,

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + ... + y_{n-1}) + 2(y_2 + y_4 + ... + y_{n-2})]$$

This is known as Simpson's one-third rule.
While using this formula, the given interval of integration must be divided into an even number of sub-intervals, since we find the area over two sub-intervals at a time.

Simpson's Three-Eight Rule (n = 3).

Putting n = 3 in formula (1) and taking the curve through (x_0, y_0), (x_1, y_1), (x_2, y_2) and (x_3, y_3) as a polynomial of degree three so that differences of order higher than two vanish, we get,

$$\int_{x_0}^{x_0+3h} f(x)dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right)$$

$$= \frac{3h}{8}[8y_0 + 12(y_1 - y_0) + 6(y_2 - 2y_1 + y_0) + (y_3 - 3y_2 + 3y_1 - y_0)]$$

$$= \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3]$$

Similarly, $\int_{x_0+3h}^{x_0+6h} f(x)dx = \frac{3h}{8}[y_3 + 3y_4 + 3y_5 + y_6]$, ...

$$\int_{x_0+(n-3)h}^{x_0+6h} f(x)dx = \frac{3h}{8}[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$

Adding the above integrals, we get,

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + ... + y_{n-2} + y_{n-1} + 2(y_3 + y_6 + ... + y_{n-3})]$$

This is known as Simpson's three-eighth rule.
While using this formula, the given interval of integration must be divided into sub-intervals whose number *n* is a multiple of 3.

Errors in Quadrature Formula:

If y_p is a polynomial representing the function $y = f(x)$ in the interval $[a, b]$ then error in the *quadrature* formulae is given by

$$E = \int_a^b f(x)dx - \int_a^b y_p dx \tag{2}$$

Error in Trapezoidal Rule:

Expanding $y = f(x)$ in the neighborhood of $x = x_0$ by Taylor's series, we get

$$y = y_0 + (x - x_0)y_0^{'} + \frac{(x - x_0)^2}{2!}y_0^{''} + ... \tag{3}$$

$$\int ydx = \int_{x_0}^{x_0+h}\left[y_0 + (x - x_0)y_0^{'} + \frac{(x - x_0)^2}{2!}y_0^{''} + ...\right]dx$$

$$= hy_0 + \frac{h^2}{2!}y_0^{''} + \frac{h^3}{3!}y_0^{'''} + ... \tag{4}$$

Now, area of the first trapezium in the interval $[x_0, x_1] = A_1 = \frac{h}{2}(y_0 + y_1)$ (5)

Putting $x = x_0 + h,\ \ y = y_1$ in (4.3),

$$y_1 = y_0 + hy_0^{'} + \frac{h^2}{2!}y_0^{''} + ... \tag{6}$$

From (5) and (6), we get

$$A_1 = \frac{h}{2}\left[y_0 + y_0 + hy_0^{'} + \frac{h^2}{2!}y_0^{''} + ...\right] = hy_0 + \frac{h^2}{2!}y_0^{'} + \frac{h^3}{22!}y_0^{''} + ...(7)$$

Subtracting eqn. (7) from eqn. (4) gives the error in (x_0, x_1),

$$\int ydx - A_1 = \left(\frac{1}{3!} - \frac{1}{2\,2!}\right) h^3 y_0'' + \ldots\ldots = -\frac{h^3}{12} y_0'' \quad \text{Neglecting other terms}$$

Similarly, the error in $[x_1, x_2]$ is $-\frac{h^2}{12} y_1''$ and in $[x_{n-1}, x_n]$ is $\frac{-h^3}{12} y_{n-1}''$.

Hence the total error is

$$E = \frac{-h^3}{12}(y_0'' + y_1'' + \ldots\ldots + y_{n-1}'')$$

Let $y''(\xi), a < \xi < b$ be the maximum of $|y_0''|, |y_1''|, \ldots\ldots\ldots, |y_{n-1}''|$ then, we have

$$E < \frac{-nh^3}{12} y''(\xi) = -\frac{(b-a)}{12} h^2 y''(\xi)$$

$$b - a = nh$$

Hence the error in the trapezoidal rule is of order h^2.

Error in Simpson's 1/3rd Rule:

Integrating eqn. (4.3) with respect to x between the limits x_0 and x_2.

$$\int_{x_0}^{x_2} ydx = \int_{x_0}^{x_0+2h} \left[y_0 + (x - x_0) y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \ldots\ldots \right] dx$$

$$= 2hy_0 + 2h^2 y_0' + \frac{8h^3}{3!} y_0'' + \frac{16h^4}{4!} y_0''' + \frac{32h^5}{5!} y_0^{(iv)} + \ldots.. \tag{8}$$

Now, $$A_1 = \frac{h}{3}(y_0 + 4y_1 + y_2) \tag{9}$$

Where A_1 is the area of the curve in the interval $[x_0, x_2]$.

Putting $x = x_0 + h,\ y = y_1$ in (2), we get

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots.. \tag{10}$$

Putting $x = x_0 + 2h,\ \ y = y_2$ in (2), we get

$$y_2 = y_0 + 2hy_0' + \frac{4h^2}{2!} y_0'' + \frac{8h^3}{3!} y_0''' + \ldots.. \tag{11}$$

Substituting eqns. (10) and (11) in eqn. (4.9), we get

$$A_1 = 2hy_0 + 2h^2 y_0' + \frac{4h^3}{3} y_0'' + \frac{2h^4}{3} y_0''' + \frac{5h^5}{18} y_0^{(iv)} + \ldots. \tag{12}$$

Now, the error in interval $[x_0, x_2]$ is given by

$$\int_{x_0}^{x_2} ydx - A_1 = \left(\frac{4}{15} - \frac{5}{18}\right) h^5 y_0^{(iv)} + \ldots.$$

$$= -\frac{h^5}{90} y_0^{(iv)} \qquad \text{Neglecting terms of order } h^6, h^7, \ldots$$

Similarly, the principal part of error in interval $[x_2, x_4]$ is $= -\frac{h^5}{90} y_2^{(iv)}$ and so on.

Hence total principal error is

$$E = -\frac{h^5}{90}[y_0^{(iv)} + y_2^{(iv)} + \ldots. + y_{2(n-1)}^{(iv)}]$$

Let $y^{(iv)}(\xi)$ be the maximum of $| y_0^{(iv)} |, \ | y_2^{(iv)} |, \ldots.., | y_{2n-1}^{(iv)} |$

Then, we have, $E < -\frac{h^5}{90} y^{(iv)}(\xi) = \frac{-(b-a)h^4}{180} y^{(iv)}(\xi)$

Hence, the error in the Simpson’s (1/3)rd rule is or order h^4.

Note.

Similarly, the principal part of the error for Simpson's (3/8)th rule is $-\frac{3h^5}{80}y^{(iv)}$ in the interval $[x_0, x_3]$.

Romberg Integration.

In numerical analysis, Romberg's method (Romberg 1955) is used to estimate the definite integral.

$$\int_a^b f(x)dx$$

by applying Richardson extrapolation (Richardson 1911) repeatedly on the trapezium rule or the rectangle rule (midpoint rule). The estimates generate a triangular array. Romberg's method is a Newton-Cotes formula -it evaluates the integrand at equally-spaced points. The integrand must have continuous derivatives, though fairly good results may be obtained if only a few derivatives exist. If it is possible to evaluate the integrand at unequally-spaced points, then other methods such as Gaussian quadrature and Clenshaw–Curtis quadrature are generally more accurate. The method is named after Werner Romberg (1909–2003), who published the method in 1955. Now we are applying the Romberg's method of integration for the results obtained by the above method we are finding a good result comparison to above method.

However, Romberg used a recursive algorithm for the extrapolation as follows:

The estimate of the true error in the trapezoidal rule is given by

$$E_t = -\frac{(b-a)^3}{12n^2}\frac{\sum_{i=1}^{n} f''(\xi_i)}{n}$$

Since the segment width, h, is given by

$$h = \frac{b-a}{n}$$

$$E_t = -\frac{h^2(b-a)}{12}\frac{\sum_{i=1}^{n} f''(\xi_i)}{n} \qquad (13)$$

The estimate of true error is given by

$$E_t \approx Ch^2 \qquad (14)$$

It can be shown that the exact true error could be written as

$$E_t = A_1h^2 + A_2h^4 + A_3h^6 + \ldots \qquad (15)$$

and for small h,

$$E_t = A_1h^2 + O(h^4) \qquad (16)$$

Since we used $E_t \approx Ch^2$ in the formula (Equation (16)), the result obtained from Equation (14) has an error of $O(h^4)$ and can be written as

$$(I_{2n})_R = I_{2n} + \frac{I_{2n} - I_n}{3}$$

$$= I_{2n} + \frac{I_{2n} - I_n}{4^{2-1} - 1} \qquad (17)$$

Where the variable TV is replaced by $(I_{2n})_R$ as the value obtained using Richardson's extrapolation formula. Note also that the sign $\approx$ is replaced by the sign =.

Hence the estimate of the true value now is

$$TV \approx (I_{2n})_R + Ch^4$$

Determine another integral value with further halving the step size (doubling the number of segments),

$$\left(I_{4n}\right)_R = I_{4n} + \frac{I_{4n} - I_{2n}}{3} \tag{18}$$

Then

$$TV \approx \left(I_{4n}\right)_R + C\left(\frac{h}{2}\right)^4$$

From Equation (4.24) and (4.25),

$$TV \approx \left(I_{4n}\right)_R + \frac{\left(I_{4n}\right)_R - \left(I_{2n}\right)_R}{15}$$

$$= \left(I_{4n}\right)_R + \frac{\left(I_{4n}\right)_R - \left(I_{2n}\right)_R}{4^{3-1} - 1} \tag{19}$$

The above equation now has the error of $O\left(h^6\right)$. The above procedure can be further improved by using the new values of the estimate of the true value that has the error of $O\left(h^6\right)$ to give an estimate of $O\left(h^8\right)$.

Based on this procedure, a general expression for Romberg integration can be written as

$$I_{k,j} = I_{k-1,j+1} + \frac{I_{k-1,j+1} - I_{k-1,j}}{4^{k-1} - 1},\ k \geq 2 \tag{20}$$

The index k represents the order of extrapolation. For example, $k = 1$ represents the values obtained from the regular trapezoidal rule, $k = 2$ represents the values obtained using the true error estimate as $O\left(h^2\right)$, etc. The index j represents the more and less accurate estimate of the integral. The value of an integral with a $j + 1$ index is more accurate than the value of the integral with a j index.

For $k = 2$, $j = 1$,

$$I_{2,1} = I_{1,2} + \frac{I_{1,2} - I_{1,1}}{4^{2-1} - 1}$$

$$= I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15} \qquad (21)$$

Application in engineering problem.

Problem: A cross section of a racing sailboat is shown in fig .4.1(a). Wind forces (f) exerted per foot of must from the sails very as a function of distance above the deck of the boat (z),as in Fig.4.1(b). Calculate the tensile force T in the mast support cable, assuming that the right support cable in completely slack and the must joins the deck in a manner that transmits horizontal or vertical forces but on moments. Assume remains vertical.

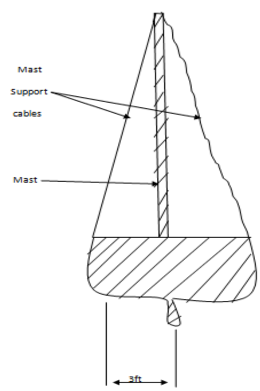
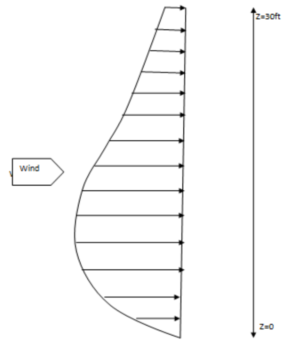
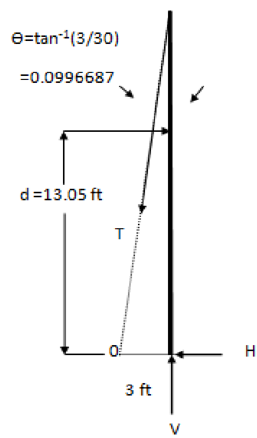


Fig (4.1) (a) cross section of a racing sailboat



(b)

Fig(4.1) wind forces (f) exerted per foot of must from the sails very as a function of distance above the deck of the boat



Fig(4.2)Forces exerted on the mast of a sailboat

Solution: To proceed with this problem, it is required that the distributed force f be converted to an equivalent total force F and that its effective location above the dack d be calculated Fig(4.2). This computation be completed by the fact that the force exerted per foot of mast varies with the distance above the deck. The total force exerted on the mast can be expressed as the integral of a continuous function.

$$F = \int_0^{30} 200\left(\frac{z}{5+z}\right)e^{-2z/30}dz$$

The nonlinear integral is difficult to evaluate analytically. Therefore, it is convenient to employ numerical approaches such as Simson's rule and the trapezoidal rule for this problem.This is accomplished by calculating f(z) for various values of z.

Description: values of f(z) for a step size 3 ft that provide data for the trapezoidal rule and Simpson's 1/3 rule.

Z,ft	0	3	6	9	12	15	18	21	24	27	30
f(z),lb/ft	0	61.40	73.13	70.56	63.43	55.18	47.14	39.83	33.42	27.89	23.20

Solution: Value of F computed on the basis of various version of the trapezoidal rule and Simpson's 1/3 rule.

Table(6) Trapezoidal Rule

Step size, ft	Segments	F,lb
15	2	1001.7
10	3	1222.3
6	5	1372.3
3	10	1450.8
1	30	1477.1
.5	60	1479.7
.25	120	1480.3
.1	300	1480.5
.05	600	1480.6

Table(7) Simpson 1/3 Rule

Step size, ft	Segments	F,lb
15	2	1219.6
5	6	1462.9
3	10	1476.9
1	30	1480.5
0.5	60	1480.6

Now we are applying the Romberg's method of integration for the results obtained by the above methods.

Result obtained by Romberg's method of integration:

I(h)=1477.1	I(h/2)=1479.7	I(h/4)=1480.3
I(h,h/2)=1477.3	I(h/2,h/4)=1480.86	I(h,h/2,h/4)=1481.34

Conclusion

We have seen that in situations where it is impossible to know the function governing some phenomenon exactly, it is still possible to derive a reasonable estimate for the integral of the function based on data points. The idea is to choose a model function going through the data points and integrate the model function. The definition of an integral as a limit of Riemann sums shows that if we chose enough data points, the integral of the model function converges to the integral of the unknown function; so theoretically, numerical integration is on solid ground. We have also seen that there are many practical factors that influence how well numerical integration works. Simple model functions may not emulate the behavior of the unknown function well. Complicated model functions are hard to work with. Problems with the number of data points, or the way in which the data was collected can have a major impact, and while we have explored some simple ways of estimating how accurate a particular numerical integral will be, this can be quite complicated in general.

On the basis of the above discussions we conclude that the result obtained by Romberg method is better than any other methods used in the study since Romberg method always modifies the results.

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